Applications of motivic homotopy theory to arithmetic counts of lines

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Abstract

A recent article by Kass and Wickelgren [9] provides a generalization of a classical theorem in algebraic geometry regarding the fact that the number of lines on a smooth projective cubic surface is always 27. Such generalization is mainly based on the study of Euler numbers in $\mathbb{A}^1$-homotopy theory. This article is one of many interesting examples where Motivic homotopy theory plays an important role in algebraic geometry. We will present a review of the techniques used in the mentioned article to achieve such generalization.

Introduction

Motivic homotopy theory was developed by Morel and Voevodsky in the '90s as a way to transport tools from algebraic topology to algebraic geometry [15]. Roughly speaking they were able to develop and use techniques from algebraic topology in order to do homotopy theory over schemes. The theory itself contains a substantial amount of advanced techniques in category theory, algebraic geometry and algebraic topology only for its construction, but the applications are surprising. Just to mention one, in 1997 Voevodsky presented a proof of Milnor's conjecture using motivic cohomology [18].

In recent years many mathematicians have focused on looking for more applications of Motivic homotopy theory, and they have been able to prove theorems and provide generalizations for different classical theorems by interpreting the classical notions in this new context. In this work we will review a particular application made by Kass and Wickelgren regarding the count of lines on smooth cubic surfaces; here is important to remark that in the 19th century, Salmon and Cayley shown that the count of lines for smooth cubic surfaces over $\mathbb{C}$ is always 27. There is a similar result when the surfaces are considered over $\mathbb{R}$ proved by Okonek and Teleman [16] and also by Finashin and Kharlamov [5], however unlike the classical result of Salmon and Cayley, there the lines are classified either as hyperbolic or elliptic and they proved that a signed count of these is always equal to 3.

Kass and Wickelgren generalized this result of Okonek and Teleman to case where the smooth cubic is considered over any field, the techniques used by them copy in somehow those of Okonek and Teleman for the real case. The main strategy of both groups of authors is to use the notion of Euler number for algebraic vector bundles. A detailed description of the methods and a modern proof of the classical theorem is contained in section 1.1. In the rest of the document we will focus on the general aspects to achieve the generalization and we will provide proofs for the fundamental aspects. This work is mainly based on [9] and [10] and for technical aspects we will refer to them. Our main goal is to provide the global idea behind the generalization and highlight the importance of the tools used as they have begun to open new ways to answer classic questions in algebraic and enumerative geometry.

Contents

Chapter 1

Arithmetic counts of lines

1.1 Motivation

We start by recalling that a cubic surface X over the complex numbers $\mathbb{C}$ is classically defined as the zero set of a cubic polynomial $f \in \mathbb{C}[x,y,z]$, that is,

$$X = \{(x,y,z) \in \mathbb{C}^3 \; : \; f(x,y,z) = 0\}$$

In modern language, X correspond to an affine scheme over $\mathbb{C}$ and we say X is smooth if is well approximated by affine spaces near each point, namely, for each point of p of X the ring $\mathcal{O}_{X,p}$ is a regular ring.
For affine varieties over an algebraically closed field, the last notion is equivalent to that the rank of the Jacobian matrix of the polynomials defining X equals the codimension of X. Recall that by compactification, X can be seen as a subscheme of $\mathbb{P}^3$ determined by a homogeneous polynomial of degree 3 in $\mathbb{C}[x,y,z,w]$. In 1849, Salmon and Cayley proved the following result regarding the number of lines in a smooth cubic surface:

Theorem 1.1. *Let X be a smooth projective cubic surface over $\mathbb{C}$. Then X contains exactly 27 lines.*

Proofs of this celebrated theorem can be found in [8] or a more modern proof in [4]. In 2017, Kass and Wickelgren proved a generalization of this classical theorem using elementary theory of Euler numbers in $\mathbb{A}^1-$ homotopy theory for algebraic vector bundles. In this section we will review the modern proof from [4] of the classical theorem and the next sections will be dedicated to study the generalization made by Kass and Wickelgren. We start with an example:

Proposition 1.2. *Consider the Fermat cubic X in $\mathbb{P}^3$, defined as the projective zero set of the polynomial*

$$f(x,y,z,w) = x^3 + y^3 + z^3 + w^3.$$

Then X contains exactly 27 lines.

Proof. Recall that in this context a line is just a 1-dimensional projective linear subspace of $\mathbb{P}^3$. Notice that a up to a permutation of coordinates, every line in $\mathbb{P}^3$ is given by linear equations of the form:

$$x = az + bw$$
$$y = cz + dw$$

for certain $a, b, c, d \in \mathbb{C}$. Such a line is in X if and only if

$$(az + bw)^3 + (cz + dw)^3 + z^3 + w^3 = 0$$

By comparing the coefficients in the expansion of the last equation, equivalently we get:

$$a^3 + c^3 = -1 \tag{1-1}$$
$$b^3 + d^3 = -1 \tag{1-2}$$
$$a^2 b = -c^2 d \tag{1-3}$$
$$ab^2 = -cd^2 \tag{1-4}$$

If a, b, c, d are different from zero, the square of the third expression divided by the fourth gives $a^3 = -c^3$ which contradicts (1-3). Hence for a line to lie in X at least one of the coefficients a, b, c, d must be zero. After a possible renumbering of the coordinates we may assume $a = 0$. In this case we get that $c^3 = -1$ and $d = 0$, and $b^3 = -1$. For such values the above equations all hold and therefore this in fact determine a line in the cubic.
In this way we obtain nine lines in X by setting $c = -\omega^j$ and $b = -\omega^k$ where $j, k = 0, 1, 2$ and ω is a primitive third root of unity. By allowing permutations of the coordinates we get $\frac{\binom{4}{2}}{2} * 3 * 3 = 27$ lines on X:

$$x + w\omega^k = y + z\omega^j = 0$$
$$x + z\omega^k = w + y\omega^j = 0$$
$$x + y\omega^k = w + z\omega^j = 0$$

where $j, k = 0, 1, 2$. □

Now we will study a modern proof of Theorem 1.1 based on [4]. Before doing that we will review some of the theory about Euler classes in vector bundles, This notion will allow us to count the zeros of a section in a given vector bundle. We will use some facts about orientations in vector bundles which we will study later for the proof of the generalization of Theorem 1.1 made by Kass and Wickelgren.

Let $E \longrightarrow M$ be a rank r oriented (relatively oriented resp.) vector bundle on a (real resp.) complex manifold M and σ be a section with only isolated zeros, this means that for every $p \in M$ with $\sigma(p) = 0$, there is an open neighborhood U of p such that the only zero of σ in U

is p.

Consider the degree of a map between oriented topological spheres $\mathbb{S}^d \longrightarrow \mathbb{S}^d$. The degree only depends on the homotopy class of the map (see [13], pag 28), namely we have a map:

$$deg : [\mathbb{S}^d, \mathbb{S}^d] \longrightarrow \mathbb{Z}$$

Now, let p be a zero of σ. By hypothesis p is isolated, thus there are local coordinates around p and a local trivialization of E (compatible with the relative orientation) such that the section σ can be identified with a function $\mathbb{C}^n \longrightarrow \mathbb{C}^n$ which we will denote also by σ (resp. $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$). Such coordinates can be chosen in a such a way that p corresponds to the origin and then there is a ball B sufficiently small around the origin such that p is the only zero of σ in B. By using σ and B we obtain a function between oriented spheres:

$$\hat{\sigma} : \partial B \longrightarrow \partial\{x \in \mathbb{C}^n \ : \ ||x|| = 1\}$$

given by $\hat{\sigma}(x) = \frac{\sigma(x)}{||\sigma(x)||}$. We then define the local degree of σ at p to be

$$\deg_p \sigma = \deg(\hat{\sigma})$$

where the expression on the right is the classical usual degree in algebraic topology. We then define the Euler number of E, denoted by $e(E)$ by means of:

$$e(E) = \sum_{p \ : \ \sigma(p)=0} \deg_p \sigma.$$

The Poincaré-Hopf theorem shows that the Euler number of E is independent of the choice of σ. Now we will review the proof of the Salmon-Cayley theorem.

Proof. Let $\mathrm{Gr}(1,3)$ be the Grassmannian parametrizing 2-dimensional subspaces W of a 4-dimensional vector space $\mathbb{C}^4$, or equivalently lines in $\mathbb{P}^3$. Denote the tautological bundle of $\mathrm{Gr}(1,3)$ by $\mathbf{S} \longrightarrow \mathrm{Gr}(1,3)$ whose fiber over a subspace W is W itself and recall that the third symmetric power of the dual of $\mathbf{S}$, denoted by $\mathrm{Sym}^3(\mathbf{S}^*)$, is also a vector bundle over $\mathrm{Gr}(1,3)$ and its fiber over the point corresponding to W is the space of cubic homogenous polynomials on W, namely $\mathrm{Sym}^3(W^*)$. Since X is defined by a cubic homogeneous polynomial on the whole 4 dimensional vector space $\mathbb{C}^4$, f determines an element of each fiber by restriction. That is, we can define a section σ_f of $\mathrm{Sym}^3(\mathbf{S}^*)$ by setting:

$$\sigma_f(W) = f|_W.$$

It can be proved that a line $\mathbb{P}(W)$ is contained in X when f vanishes on W or in other words, there is a bijection between the zeros of the section σ_f and the lines in X. We will see later that every zero L of σ_f is isolated and the local degree $\deg_L \sigma_f$ is equal to 1. As a consequence the Euler number of $\mathrm{Sym}^3(\mathbf{S}^*)$ gives a count of the lines in the cubic smooth surface X. In particular this number is independent of the surface (because does not depend on the section σ_f) and therefore Proposition 1.2 completes the proof. □

Similar techniques have been used to prove the last result in the case where the surface is defined over $\mathbb{R}$, however in this case the count does not give 27 lines, instead, one can define the type of a line and classify them as hyperbolic and elliptic lines. The problem here is that the number of lines depends on the surface but a certain signed count does not. This assertion was proved by Okonek and Teleman in [16] and also by Finashin-Kharlamov in [5].The result is as follows:

Theorem 1.3. *Let X be a smooth cubic surface over $\mathbb{R}$. If h denotes the number of hyperbolic lines in X and e the number of elliptic lines. Then $h-e=3$.*

Motivated by the techniques used to prove theorems 1.1 and 1.3, Kass and Wickelgren observed a general principle and proved a generalization over any arbitrary field k. To this end, the authors used the $\mathbb{A}^1$-homotopy theory developed by Morel and Voevodsky [15] and in particular the definition made by Morel of the degree homomorphism:

$$\deg : [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \longrightarrow \mathrm{GW}(k).$$

Here the expression on the right is the Grothendieck-Witt group of bilinear forms over a field. The definition of this degree homomorphism, sometimes called local degree in $\mathbb{A}^1$-homotopy theory requires a very good understanding of Morel-Voevodsky's theory, however the calculations made in [9] are performed in an elementary way and without direct reference to $\mathbb{A}^1$-homotopy theory. We will study the general idea behind the proof and for technical aspects we will refer to [9]. Furthermore, we will assume some results regarding the $\mathbb{A}^1$-degree in order to focus on the generalization of Salmon-Cayley's theorem and not on the general aspects of this immense but fascinating theory.

1.2 Basic definitions and properties

We start by setting some notation and recalling some definitions and properties about Grassmannians that we used before. Given a k-vector space A, the Grassmannian $\mathrm{Gr}(A, r)$ is defined as the set of all r-dimensional subspaces of A. We recall that if $B \subseteq A$ is a r-dimensional linear subspace of A spanned by the vectors $v_1, ..., v_r$, we can associate to B the multivector:

$$\lambda = v_1 \wedge \cdots \wedge v_r \in \Lambda^r(A).$$

The multivector λ is determined up to scalars by B. If we choose a different basis, the corresponding λ would simply be multiplied by the determinant of the base-change matrix, thus we have a well defined map (so far as sets):

$$\psi : \mathrm{Gr}(A, r) \longrightarrow \mathbb{P}(\Lambda^r A)$$

This map is in fact an inclusion because for any $[w] \in \psi(B)$ in the image, we can recover B as the space of vectors $v \in A$ such that $v \wedge w = 0 \in \Lambda^{r+1} A$. This embedding is called Plücker

embedding and is well studied for instance in [7]. This embedding describes $G(A,r)$ as a subvariety of $\mathbb{P}(\Lambda^r A)$. In modern language the Grassmannian can be constructed as a scheme by expressing it as a representable functor. We will assume this fact and work sometimes with this scheme structure for $\mathrm{Gr}(A,r)$.
To fix notation, we will write $\mathrm{Gr}(n,r)$ for $\mathrm{Gr}(k^n,r)$ and $\mathbb{P}(A)$ is just $\mathrm{Gr}(A,1)$ (equivalently this can be seen as $\mathrm{Proj}(\mathrm{Sym}(A^*))$. With this notation, the global sections $H^0(\mathbb{P}(A),\mathcal{O}(1))$ are identified with the linear functionals on A, that is, A^*. Here $\mathcal{O}(1)$ denotes the classical twisting sheaf of Serre associated to the projective variety $\mathbb{P}(A)$.

One important notion regarding the study and understanding of the geometry of Grassmannians is the notion of *Universal bundles.* We will review the fundamental definitions and state, but not proof, a correspondence between scheme morphisms, with $G(A,r)$ as target and certain kind of subbundles. This theorem will be relevant to study some properties of lines in the Grassmannian.
As before, let A be a k-vector space of dimension n and set $G = \mathrm{Gr}(A,r)$ the Grassmannian of r-dimensional vector subspaces of A. The trivial vector bundle of rank n on G is just given by $\mathcal{V} := G\times A$, its fiber at each point is the vector space A. Here it is important to recall that vector bundles and locally free sheaves are virtually the same, so we will think the vector bundle as a variety rather than a locally free sheaf.
Let $\mathcal{S}$ be the subbundle of $\mathcal{V}$ of rank r whose fiber at each point $[W]\in G$ is the subspace W itself; namely,

$$\mathcal{S}_{[W]} = W \subseteq A = \mathcal{V}_{[W]}$$

$\mathcal{S}$ is called the *universal subbundle* on G and the quotient bundle $\mathcal{Q} = \mathcal{V}/\mathcal{S}$ is called the *universal quotient bundle.* In the case where $r = 1$ we have that $G = \mathbb{P}(A) \cong \mathbb{P}^{n-1}$ and the universal subbundle is just given by the dual of the Serre's twisting sheaf $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ which is the line bundle $\mathcal{O}_{\mathbb{P}(A)}(-1)$. Furthermore, in the case $r = n-1$, the identification $\mathrm{Gr}(A,r) = \mathrm{Gr}(A^*, n-r)$ gives that $G = \mathbb{P}(A^*)$ and so the universal quotient bundle Q is the line bundle $\mathcal{O}_{\mathbb{P}(A^*)}(1)$. It can be proved that these last $\mathcal{S}$ and $\mathcal{Q}$ are in fact vector bundles over G, see for instance [4, Section 3.2.3]. Now we state the relevant correspondence:

Theorem 1.4 ([4])**.** *Let X be any scheme. Then morphisms $\phi : X \longrightarrow \mathrm{Gr}(A,r)$ are in one-to-one correspondence with rank r subbundles $\mathcal{F} \subseteq A\otimes \mathcal{O}_X$. Explicitly, ϕ corresponds to the bundle $\mathcal{F} = \phi^*\mathcal{S}$.*

Definition 1.5. Let X be a k-projective variety. A *linear system* on X is a pair $(T,\mathcal{L})$ consisting of a line bundle (invertible sheaf) $\mathcal{L}$ and a k-vector subspace T of the global sections $H^0(X,\mathcal{L})$. If $T = H^0(X,\mathcal{L})$ then the linear system is called *complete.* Usually we will refer just to the subspace T letting the other data implicit.

A point $p \in X$ on which all elements of the linear system T vanish is called a *base point* of the linear system. If T has no base points we say that it is *base point-free*. The set of base points is called the *base locus* of the linear system and it is a closed subset of X. Taking the scheme-theoretic intersection we can define the *scheme-theoretic base locus* or just *base scheme*. Notice that the linear system T is base point-free if $\cap_{s\in T}\{s=0\}$ is the empty subscheme.

A very well known theorem in algebraic geometry states that there is a correspondence between invertible sheafs and maps to a projective space. Specifically:

Theorem 1.6. *Let A be a ring and X be a scheme over A. Then if $\phi : X \longrightarrow \mathbb{P}^n_A := ProjA[x_0,...,x_n]$ is a morphism of schemes, the pullback of the invertible sheaf $\mathcal{O}(1)$ under ϕ, $\phi^*(\mathcal{O}(1))$, is also an invertible sheaf on X and is generated by the global sections $s_i = \phi^*(x_i)$, $i = 0,1,...,n$. Conversely, for any invertible sheaf $\mathcal{L}$ and for all global sections $s_1,...,s_l$ that generate L there is a unique morphism $\phi : X \longrightarrow \mathbb{P}^n_A$ such that $\mathcal{L} \cong \phi^*(\mathcal{O}(1))$ and $s_i = \phi^*(x_i)$ under the isomorphism.*

Proof. [8, II-Theorem 7.1]. □

A coordinate free version of the last theorem then state that any (finite dimensional) base-point-free system $(T,\mathcal{L})$ on X induces a morphism to a projective space

$$\phi_T : X \longrightarrow \mathbb{P}(T^*)$$

where $\phi_T^* \cong \mathcal{L}$.

In what follows we will define what we mean by a line in a scheme and define the type of a line.

Definition 1.7. A *line* l in $\mathbb{P}^3_k = \mathrm{Proj}(k[x_0,...,x_3])$ is defined to be a closed point in the variety $\mathrm{Gr}(4,2)$. The residue field of a closed point will be called the *field of definition* of l. Recall that the residue field at a closed point on a scheme is defined as the the residue field of the corresponding stalk at the point of the structure sheaf, which is by definition a local ring.

Remark 1.8. To any line l with field of definition L we can associated a closed subscheme of $\mathbb{P}^3_k$ in the following way: the closed point $l \in \mathrm{Gr}(4,2)$ defines a morphism $\mathrm{Spec}(L) \longrightarrow \mathrm{Gr}(4,2)$. Theorem 1.4 allow us to associate a rank 2 subbundle $S \subseteq k^4 \otimes L \cong L^4$ (since L is a field, the structural sheaf of $\mathrm{Spec}(L)$ is just L itself). Then the homogeneous ideal generated by $\mathrm{ann}(S) \subseteq \mathrm{Sym}((k^4)^*)$ defines a subscheme of $\mathbb{P}^3_k$. By abuse of notation we will denote this subscheme with the same letter as the point l.

Interlude: Grothendieck-Witt group

The Grothendieck-Witt group mentioned in the motivation is a sufficiently complicated group to support some invariants and is simple enough to make computations.

Let k be a field and $M(k)$ be the set of isomorphism classes of non singular quadratic forms over k. It can be shown that $M(k)$ is a commutative semiring under the operations $\perp$ and $\otimes$ (see

[11]). The additive structure $\perp$ makes $M(k)$ into a monoid with the cancellation property thus performing a Grothendieck group construction (also called group completion), we obtain a group denoted by $\mathrm{GW}(k)$ which is just $\mathrm{Gro}(M(k))$. For an element $a \in k \setminus \{0\}$ and for any $x, y \in k$, the element of the Grothendieck-Witt group represented by the symmetric, non singular rank 1 bilinear form defined by $(x, y) \mapsto axy$ will be denoted by $\langle a \rangle$.

1.3 Hyperbolic and elliptic lines

Let k be a field such that $\mathrm{char}(k) \neq 2$ and fix a cubic polynomial $f \in k[x_0, x_1, x_2, x_3]$ that defines a smooth cubic surface over k, namely $X := \{[p] \in \mathbb{P}^3_k \ : \ f(p) = 0\}$.

Definition 1.9. If l is a line in X with field of definition L, we define $T \subseteq (L^4)^* = H^0(\mathbb{P}^3_L, \mathscr{O}_X(1))$ to be the vector space of linear polynomials that vanish on l.

The last definition could be not very enlightening, however it can be proved that it coincides with the classical definition of Segre (see [5]). On the other hand, notice that T is canonically a subspace of the vector space of global sections of $\mathscr{O}_X(1)$, moreover if we denote the ideal sheaf of the closed subscheme l by I_l, T is also a subspace of the space of global sections of the sheaf $I_l \cong I_l \otimes \mathscr{O}(1)$ of linear polynomials vanishing on l. T can be alternatively described as $\mathrm{ann}(S)$ for $S \subseteq L^4$.
The ideal sheaf I_l is a line bundle because l is a codimension 1 closed subscheme of $X \otimes L$ (where $X \otimes L$ denotes the subscheme obtained from X by base change from k to L, in some books is denoted by X_L). Thus we have defined two linear systems on X: $(T, \mathscr{O}(1))$ and $(T, I_l(1))$. It turns out that $(T, \mathscr{O}(1))$ has base points (the points of l) but the other system is base-point-free.

Lemma 1.10. *The linear system $(T, I_l(1))$ considered before is a base-point-free system.*

Proof. Proving that $(T, I_l(1))$ is base-point-free is equivalent to show that the sheaf $I_l(1)$ is globally generated. Notice that $I_l(1)$ is the restriction of the sheaf on $\mathbb{P}^3_L$ and by definition of the subscheme l, it is generated by T. In consequence the same holds for X and therefore $I_l(1)$ is globally generated. □

Remark 1.11. We have seen that any base-point-free system induces a morphism to a projective space. Let's denote by $\pi : X \otimes L \longrightarrow \mathbb{P}(T^*)$ the morphism associated to $(T, I_l(1))$. The restriction of π to the closed subscheme l is a finite morphism of degree 2 (here the degree is regarded as the degree of the corresponding field extension on function fields). Since $\mathrm{char}(k) \neq 2$, the extension is therefore Galois (any degree 2 extension of fields is Galois). The Galois group then has a nontrivial element of order 2. We will denote the nontrivial element of the Galois group $l \longrightarrow \mathbb{P}(T^*)$ by $i : l \longrightarrow l$. Since i has order 2, defines an involution which will play a fundamental role in the definition below.

Before stating such definition let's remark that we will use the notion made by Morel of $\mathbb{A}^1$-degree. Such definition requires a very good understanding of Motivic homotopy theory and Milnor-Witt K-theory. However when considering the degree of the involution i before, there are simpler descriptions and useful results used by Kass and Wickelgren to avoid this construction. For the reader with knowledge in this area, let us mention that Morel constructed the $\mathbb{A}^1$-degree as an isomorphism:

$$\deg_{\mathbb{A}^1} : [\mathbb{P}^1_k, \mathbb{P}^1_k]^{\mathbb{A}^1} \longrightarrow \mathrm{GW}(k)$$

where the first group is the group of (stable pointed) $\mathbb{A}^1$-homotopy classes (constructed by means of model categories) and the set on the right is the Grothendieck-Witt group $\mathrm{Gro}(M(k))$.

Definition 1.12. A line l on X is said to be *hyperbolic* if its *type* defined as the expression $\langle -1\rangle \cdot \deg_{\mathbb{A}^1}(i)$ equals $\langle 1\rangle$ in $\mathrm{GW}(k)$. In any other case l is called *elliptic*.

Remark 1.13. In the previous definition if $\mathrm{char}(k) = 2$, the involution may not exist in which case the type is undefined. We will see an example of this later.
On the other hand if we identify l with $\mathbb{P}^1_L$ then i corresponds to a linear fractional transformation of the form $\frac{az+b}{cz+d}$ for $a, b, c, d \in k$. In this case the $\mathbb{A}^1$-degree of i is equal to $\langle ad - bc\rangle \in \mathrm{GW}(k)$. In particular l being hyperbolic equals to $-(ac - bd)$ being a perfect square in L.

Before dealing with Euler numbers we will need some results about the involution i in order to get simpler expressions for the $\mathbb{A}^1$-degree of the involution.

Lemma 1.14. *Let l be a line defined by the subspace spanned by the vectors $(0,0,1,0)$ and $(0,0,0,1)$ and let P_0, P_1 quadratic homogeneous polynomials in $k[x_0, x_1, x_2, x_3]$ such that the polynomial defining X can be written as $f = x_0P_0 + x_1P_1$.*
If l lies in X, then the fiber of $\pi : l \longrightarrow \mathbb{P}(T^*)$ over the k-point corresponding to the subspace spanned by $(a, b, 0, 0) \in T^*$ is

$$\{aP_0(0, 0x_2, x_3) + bP_1(0, 0, x_3, x_4) = 0\} \subseteq l.$$

Proof. Clearly the point corresponding to $(a, b, 0, 0)$ is the zero set of the polynomial $bx_0 - ax_1$, considered as a global section of $\mathcal{O}_{\mathbb{P}(T^*)}(1)$. By the construction of $\pi : l \longrightarrow \mathbb{P}(T^*)$ is the zero locus of $bx_0 - ax_1$ considered as a global section of $\mathcal{O}_l \otimes I_l(1)$. If we can identify the last sheaf with $\mathcal{O}_l(2)$ then $bx_0 - ax_1$ can be identified with the polynomial $aP_0(0, 0x_2, x_3) + bP_1(0, 0, x_3, x_4)$. For points in X, $x_0P_0 + x_1P_1 = 0$, so $x_1 = -\frac{P_1}{P_0}x_2$ and therefore as a section, x_1 generates $I_l(1)$ on $P_1 \neq 0$. Similarly x_0 generates $I_l(1)$ on $P_2 \neq 0$. As a consequence, the analogue is true for $\mathcal{O}_l \otimes I_l(1)$, and therefore the map sending x_2 to $-P_0(0, 0, x_2, x_3)$ and x_1 to $P_1(0, 0, x_2, x_3)$ defines an isomorphism $\mathcal{O}_l \otimes I_l(1) \cong \mathcal{O}_l(2)$ that sends $bx_0 - ax_1$ to $aP_0(0, 0x_2, x_3) + bP_1(0, 0, x_3, x_4)$ and the lemma follows. □

As a consequence of the last Lemma, notice that if X is the cubic surface defined by $f = x_0^3 + x_1^3 + x_2^3 + x_3^3$ over the finite field $\mathbb{F}_2$, this surface contains the line l spanned by $(1, 1, 0, 0)$

and $(0,0,1,1)$. Therefore, the lemma implies that the morphism $\pi : l \longrightarrow \mathbb{P}(T^*)$ defines a purely inseparable extension and in particular l does not admit a nontrivial automorphism compatible with π. This shows how the assumption $\mathrm{char}(k) \neq 2$ is necessary to define the types of lines.

Some results about the involution i will allow us to get expressions for $\deg_{\mathbb{A}^1}(i)$. We summarize such results below.

Proposition 1.15. *Every nontrivial involution of the projective space $\mathbb{P}^1_k$ is conjugate to the involution $z \mapsto -\alpha/z$ for some $\alpha \in k$.*

Proof. See [1, Theorem 4.2] □

Proposition 1.16. *The $\mathbb{A}^1$-degree of the involution $i(z) = -\frac{\alpha}{z}$ is $\langle \alpha \rangle \in GW(k)$.*

Proof. See [3, Theorem 3.6] □

As a consequence of the last two propositions we have the following:

Corollary 1.17. *Let $i : \mathbb{P}^1_k \longrightarrow \mathbb{P}^1_k$ be a nontrivial involution and $D \in k$ the discriminant of the fixed subscheme associated to i, then we have that:*

$$\langle -1 \rangle \deg_{\mathbb{A}^1}(i) = \langle D \rangle \in GW(k).$$

Proof. The $\mathbb{A}^1$-degree and the class of the discriminant are invariant under conjugations [14] therefore by Proposition 1.15 is sufficient to prove the corollary for the involution $i(z) = -\frac{\alpha}{z}$. The fixed subscheme for this particular involution is then $\{z^2 + \alpha = 0\}$ and its discriminant is clearly -4α. By manipulation of classes we have that $\langle -4\alpha \rangle = \langle -\alpha \rangle$ and by proposition 1.16 the degree $\deg_{\mathbb{A}^1}(i) = \langle \alpha \rangle$ thus

$$\langle -1 \rangle \deg_{\mathbb{A}^1}(i) = \langle -1 \rangle \langle \alpha \rangle = \langle -\alpha \rangle = \langle D \rangle.$$

□

There is another interesting equality for the $\mathbb{A}^1$-degree of the involution i which will be useful later:

Lemma 1.18. *Let $e_1, .., e_4$ be a basis for k^4 and $S := span(e_3, e_4)$ be a 2-dimensional subspace of k^4 that defines a line contained in X. Then:*

$$\langle -1 \rangle \deg_{\mathbb{A}^1}(i) = \left\langle Res\left(\left. \frac{\partial f}{\partial e_1} \right|_S, \left. \frac{\partial f}{\partial e_2} \right|_S \right) \right\rangle \in GW(L)$$

Proof. Corollary 1.17 says that it is enough to compute the discriminant of the fixed locus of the involution. One can use the identification between the fixed locus and its image on the ramification locus of the morphism $\pi : l \longrightarrow \mathbb{P}(T^*)$ and compute the discriminant on the ramification locus using Lemma 1.14. l can be assumed to be the line defined by the subspace span((0,0,1,0),(0,0,0,1)) (as soon as we extend scalars and choose the appropriate change of coordinates). Write $f = x_1 P_1 + x_2 P_2$ and apply Lemma 1.14. This implies that the ramification locus is defined as the locus where

$$aP_1(0,0,x_3,x_4) + bP_2(0,0,x_3,x_4)$$

has a multiple root in x_3, x_4. As a consequence, the ramification locus is the zero set of

$$\mathrm{Disc}_{x_3,x_4}(aP_1(0,0,x_3,x_4) + bP_2(0,0,x_3,x_4)).$$

Therefore, the discriminant of the image on the ramification locus and hence, the discriminant of the fixed locus of the involution is just,

$$\mathrm{Disc}_{a,b}\left(\mathrm{Disc}_{x_3,x_4}(aP_1(0,0,x_3,x_4) + bP_2(0,0,x_3,x_4))\right) \in k/(k)^*.$$

On the other hand, if we differentiate $f = x_3 P_1 + x_4 P_2$, we get:

$$\left.\frac{\partial f}{\partial e_1}\right|_S = P_1(0,0,x_3,x_4)$$
$$\left.\frac{\partial f}{\partial e_2}\right|_S = P_2(0,0,x_3,x_4).$$

Therefore, if $P_1(0,0,x_3,,x_4) = \sum a_i x_3^i x_4^{2-i}$ and $P_2(0,0,x_3,x_4) = \sum b_i x_3^i x_4^{2-i}$, then the computation of the resultant gives:

$$\begin{aligned}\mathrm{Res}(P_1(0,0,x_3,x_4), P_2(0,0,x_3,x_4)) &= a_1^2 b_0 b_2 - a_2 a_1 b_0 b_1 - a_0 a_1 b_1 b_2 + a_2^2 b_0^2 + a_0 a_2 b_1^2 + a_0^2 b_2^2 - 2a_0 a_2 b_0 b_2 \\ &= \frac{1}{16}\mathrm{Disc}_{a,b}\left(\mathrm{Disc}_{x_3,x_4}(aP_1(0,0,x_3,x_4) + bP_2(0,0,x_3,x_4))\right)\end{aligned}$$

Since they both differ only by a number, its classes in the Grothendieck-Witt group are the same and the Lemma follows. □

1.4 Euler numbers

Here we will define Euler numbers in $\mathrm{GW}(k)$ for algebraic vector bundles that are "relatively" oriented. This notion needs to be considered because the Grassmannian $\mathrm{Gr}(4,2)$ has a non-orientable tangent bundle, however it is relatively oriented. Here X will denote a smooth r-dimensional k-scheme.

Definition 1.19. Let $\pi : E \longrightarrow X$ be a rank r algebraic vector bundle on a smooth r-dimensional k-scheme X. A *orientation* of E consists of a line bundle L and a isomorphism of the form $L \otimes L \cong \Lambda^r E$. Furthermore a *relative orientation* of E is an orientation of the vector bundle $\mathrm{Hom}(\Lambda^r TX, \Lambda^r E)$, here TX denote the tangent bundle of X.

Example 1.20. If $X = \mathbb{P}^1$ and $\mathcal{O}(-1)$ is its tautological vector bundle, then $\Lambda^{\text{top}} TX \cong TX \cong \mathcal{O}(2)$. Thus line bundles $\mathcal{O}(n)$ on $\mathbb{P}^1$ are relatively oriented if and only if n is even.

Remark 1.21. Let $\pi : E \longrightarrow X$ a relatively oriented rank r vector bundle. On an open subset U of X, a section σ of $\text{Hom}(\Lambda^r TX, \Lambda^r E)$ is called a *square* if its image under the isomorphism $\Gamma(U, \text{Hom}(\Lambda^r TX, \Lambda^r E)) \cong \Gamma(U, L \otimes L)$ is the tensor square of a section in $\Gamma(U, L) = L(U)$.

The Euler number for a vector bundle could have been defined as in 1.1. Recall that it was defined as a sum of local degrees of a section with isolated zeros, however this definition is linked to choosing trivializations of the vector bundle in order to identify the section with a function. For oriented vector bundles this works good, but if E is not oriented and if we change the local trivialization by a linear function with negative determinant this will change the sign of the local degree. That's why in 1.1 we asked for trivializations compatible with the orientations. To sum up, we must choose coordinates and trivializations that are compatible with the relative orientation. The "good" coordinates we should then consider are called Nisnevich coordinates:

Definition 1.22. Let p be a closed point of X and U be a open Zariski neighborhood of p. An étale map $\phi : U \longrightarrow \mathbb{A}^r = \text{Spec}[x_1, ..., x_r]$ is called *Nisnevich* local coordinates around p if the induced map of residue fields $k(\phi(p)) \longrightarrow k(p)$ is an isomorphism.

When $k(p)$ is a separable extension of k and $r \geq 1$, Nisnevich coordinates are guarantee to exist. This is justified thanks to the following proposition from [2]:

Proposition 1.23. *If X is a smooth curve over k and p is a closed point of X such that $k(p)$ is a separable extension of k, then there exists Nisnevich coordinates around p.*

Local coordinates around a point p give a distinguished trivialization of TX. Taking the wedge product of the basis of vector fields (sections of TX), we obtain a distinguished section in $\Lambda^r TX(U)$ for some neighborhood U of p. Similarly, a choice of local trivialization of E gives a distinguished section in $\Lambda^r E(U)$, by possibly shrinking U this gives a distinguished section of $\text{Hom}(\Lambda^r TX, \Lambda^r E)(U)$. This motivates the following definition:

Definition 1.24. Local coordinates and a trivialization of the vector bundle E on a open neighborhood U of p are compatible with the relative orientation of E if the distinguished section of $\text{Hom}(\Lambda^r TX, \Lambda^r E)(U)$ is the tensor square of a section in $L(U)$, where L denotes the line bundle coming from the orientation of E.

This provides a good definition of "compatibility" in terms of Nisnevich coordinates. Let ϕ be Nisnevich coordinates around p. Since ϕ is étale, the standard basis for the tangent space of $\mathbb{A}^r_k$ provides a local trivialization for the restriction vector bundle $TX|_U$. By potentially shrinking U we can assume that the restriction of E to U is trivial.
A trivialization of $E|_U$ is then called compatible with ϕ and the relative orientation if the distinguished section of $\text{Hom}(\Lambda^r TX|_U, \Lambda^r E|_U)$ taking the distinguished basis of $\Lambda^r TX|_U$ to the

distinguished basis of $\Lambda^r E|_U$ is a square, that is, it is the image of the tensor square of a section in $L(U)$ under the isomorphism $\mathrm{Hom}(\Lambda^r TX, \Lambda^r E) \cong L \otimes L$, where L comes from the relative orientation.

Let's fix $\phi : U \longrightarrow \mathbb{A}^r_k$ Nisnevich coordinates around p and $\psi : E|_U \longrightarrow \mathscr{O}^r_U$ be a local trivialization of E which is compatible with the relative orientation (considerations are made after possibly shrinking U). Before proving and stating some technical results let's see how we can define the local degree of a section with only "isolated" zeros (see Definition 1.25).
Let σ be a global section of E and p be an isolated zero. The composite $\psi \circ \sigma|_U$ is then an element of $\mathscr{O}^r(U)$. We wish to identify the section with a function $\mathbb{A}^r \longrightarrow \mathbb{A}^r$ that is, we want that each of the r components of $\psi \circ \sigma|_U$ to be in the image of the pullback $\phi^* : \mathscr{O}_{\mathbb{A}^r} \longrightarrow \mathscr{O}_U$. If X is covered by open sets of the form $\mathbb{A}^r$ (like the Grassmannian) the latter can be done because ϕ can be chosen to be an isomorphism on local rings. The general case requires a little more work and involves adding an element $G = (g_1, ..., g_r)$ of $\mathscr{O}^r_U$ so that $G + \psi \circ \sigma = \psi^*(F)$ for some $F : \mathbb{A}^r \longrightarrow \mathbb{A}^r$. We can then define

$$\deg_p \sigma = \deg_{\phi(p)} F.$$

In what follows we will focus on this construction and on proving that the definition of the degree is independent of the choices of ϕ, ψ and G.

As before, let $\phi : U \longrightarrow \mathbb{A}^r_k$ be Nisnevich coordinates around p and $\psi : E|_U \longrightarrow \mathscr{O}^r_U$ be a local trivialization of E. Let $r_U \in L(U)$ denote the element such that $r_U \otimes r_U$ is mapped to the distinguished section of $\mathrm{Hom}(\Lambda^r TX|_U, \Lambda^r E|_U)$ under the isomorphism provided by the relative orientation. Let σ be a global section of E and denote $Z \subseteq X$ the closed subscheme defined by the zero locus of σ, that is, $Z := \{\sigma = 0\}$.

Definition 1.25. A closed point p of X is called *isolated zero* of σ if $p \in Z$ and its local ring $\mathscr{O}_{Z,p}$ is a finite k-algebra. Moreover if $\mathscr{O}_Z$ is a finite k-algebra, we say that σ has isolated zeros.

Some authors define isolated points as points for which there exists a Zariski open neighborhood U in such a way that the set-theoretic intersection $U \cap Z$ only contains the isolated point in consideration. This is in fact an equivalence as we prove next.

Proposition 1.26. *Under the same notations of the last definition, a point p of X is an isolated zero of σ if and only if there exists a Zariski open neighborhood U of p such that $U \cap Z = \{p\}$. Furthermore, σ has isolated zeros if and only if Z consists of finitely many closed points.*

Proof. Suppose p is an isolated zero of σ, then since $\mathscr{O}_{Z,p}$ defines a finite k-algebra its dimension must be zero, and since p is closed $\mathscr{O}_{Z,p}/p$ also has dimension 0. The irreducible component of Z that contains p is then a finite type subscheme Z_0 over k, and therefore we have:

$$\dim(Z_0) = \dim(\mathscr{O}_{Z,p}) + \dim\big(\mathscr{O}_{Z,p}/p = 0\big).$$

It follows that Z_0 is irreducible, has dimension zero and is finite type over k thus it must be a single point and there is no other option than p. Define U to be the complement of the union of the other irreducible components of Z. Clearly U is Zariski open and $U \cap Z = \{p\}$. Conversely, if p is a closed point such that there is a Zariski open neighborhood U such that $U \cap Z = \{p\}$ this clearly implies that the dimension of the irreducible component of Z containing p is zero and therefore Z_0 is finite type over k (because is zero dimensional). Using the same dimensional equality we used before we can conclude that $\mathscr{O}_{Z,p}$ is a finite k-algebra and the point p is isolated. The last part of the proposition follows thanks to the fact that zero dimensional Noetherian rings have finitely many points so when σ has isolated zeros, Z has finitely many points. Using the fact that Z is zero dimensional it can be also prove that all points of Z are closed and therefore Z is a zero dimensional finite type k-algebra and therefore finite. □

Lemma 1.27. *Let p be an isolated zero of σ. Then $\mathscr{O}_{Z,p}$ is finitely generated as a k-algebra by $x_1,...,x_r$. Moreover, for any positive integer m, the local ring $\mathscr{O}_{X,p}/p^m$ is finitely generated as a k-algebra by $x_1,...,x_r$.*

Proof. We claim that it is sufficient to show the second assertion. Indeed, since p is an isolated zero, $\mathscr{O}_{Z,p}$ is finite and therefore there is a positive integer m such that p^m is the zero ideal in $\mathscr{O}_{Z,p}$. The ring $\mathscr{O}_{Z,p}$ can be seen as a quotient of $\mathscr{O}_{X,p}$, hence the first statement in the lemma is just a particular case of the second one.
Let $\phi : U \longrightarrow \mathbb{A}^r = \mathrm{Spec} k[x_1,...,x_r]$ Nisnevich coordinates around p and let $q \subseteq k[x_1,...,x_r]$ be the corresponding prime ideal such that $\phi(p) = q$. We know that ϕ induces an isomorphism on the residue field of p, that is, $k[x_1,...,x_r]/q \longrightarrow \mathscr{O}_{X,p}/p$ is an isomorphism. We claim that for any positive integer m, the map $k[x_1,...,x_r] \longrightarrow \mathscr{O}_{X,p}/p^m$ is surjective. In fact, if $m = 1$ the claim is trivial by construction. Now by induction on m, let's assume the result is true for all positive integers less than m. Then, given an element $\bar{y} \in \mathscr{O}_{X,p}/p^m$, by the inductive hypothesis there is a $\bar{y}'$ on the image such that $y - y' \in p^{m-1}$. Write $y - y'$ as $y - y' = \sum_i a_i b_i$ for some $a_i \in p^{m-1}$ and $b_i \in p$. The Nisnevich coordinates ϕ is an étale map and therefore induces an isomorphism on cotangent spaces, thus it follows that there are $a_i' \in p^{m-2}$ and $b_i' \in p$ in the image such that $a_i - a_i' \in p^{m-1}$ and $b_i - b_i' \in p^2$. As a consequence, in the quotient $\mathscr{O}_{X,p}/p^m$ we have that $\sum_i a_i b_i = \sum a_i' b_i'$. This last sum is clearly an element of the image, which means that there is an element $f \in k[x_1,...,x_r]$ such that $f \mapsto \sum a_i' b_i' = \sum_i a_i b_i = y$. □

We have seen that if p is an isolated zero of a global section σ of E, then choosing a compatible trivialization of $E|_U$ gives a local expression for σ as a r-tuple of functions $(f_1,...,f_r) : \mathbb{A}^r \longrightarrow \mathbb{A}^r$. These functions can be seen as elements of the local ring $\mathscr{O}_{X,p}$ by restriction. The local ring $\mathscr{O}_{Z,p}$ is then isomorphic to $\mathscr{O}_{X,p}/\langle f_1,...,f_r\rangle$ and there is a commutative diagram:

$$\begin{array}{ccc} k[x_1,\dots,x_r] & & \\ \downarrow & \searrow & \\ \mathcal{O}_{X,p} & \twoheadrightarrow & \mathcal{O}_{Z,p} \end{array}$$

The ring $\mathcal{O}_{Z,p}$ is finite (p is a isolated zero) and therefore there is a positive integer m such that $p^m = 0$ in the quotient $\mathcal{O}_{Z,p} \cong \mathcal{O}_{X,p}/\langle f_1,\dots,f_r\rangle$. In particular, the latter implies that $\langle f_1,\dots,f_r\rangle = \langle f_1,\dots,f_r\rangle + p^m$ in $\mathcal{O}_{X,p}$. Lemma 1.27 implies that $k[x_1,\dots,x_r] \longrightarrow \mathcal{O}_{X,p}/p^{2m}$ is surjective, thus there are polynomials $g_1,\dots,g_r$ in $x_1,\dots,x_r$ such that $g_i - f_i \in p^{2m}$. The following lemmas serve to justify the good definition of "index" for their proofs we refer to [9].

Lemma 1.28. *The sets of functions g_i and f_i defined before satisfy the following equality of ideals: for any positive integer e we have $\langle g_1,\dots,g_r\rangle^e = \langle f_1,\dots,f_r\rangle^e$ in $\mathcal{O}_{X,p}$.*

Lemma 1.29. *The inverse of the pullback under the Nisnevich coordinates ϕ satisfy that $(\phi^*)^{-1}(\langle f_1,\dots,f_r\rangle)^e = \langle g_1,\dots,g_r\rangle^e$ in $k[x_1,\dots,x_r]_q$ for all positive integers e.*

Lemma 1.30. *There is an isomorphism $\mathcal{O}_{Z,p} \cong k[x_1,\dots,x_r]_q/\langle g_1,\dots,g_r\rangle$.This quotient is a finite complete intersection ring.*

Last isomorphism determines another isomorphism

$$\mathrm{Hom}_k(\mathcal{O}_{Z,p},k) \cong \mathcal{O}_{Z,p}$$

of $\mathcal{O}_{Z,p}$-modules. Denote by η the element of $\mathrm{Hom}_k(\mathcal{O}_{Z,p},k)$ which is the image of the unit 1 in $\mathcal{O}_{Z,p}$. A detailed discussion and justification of the existence of such isomorphism can be found in [17].

Proposition 1.31. *η is independent of the choice of $g_1,\dots,g_r$.*

Remark 1.32. Associated to η we have a symmetric bilinear form β on $\mathcal{O}_{Z,p}$ defined by

$$\beta(x,y) = \eta(xy).$$

Since the map $y \mapsto \eta(xy)$ in $\mathrm{Hom}_k(\mathcal{O}_{Z,p},k)$ is send it to x in $\mathcal{O}_{Z,p}$ under the isomorphism $\mathrm{Hom}_k(\mathcal{O}_{Z,p},k) \cong \mathcal{O}_{Z,p}$, it follows that β is non-degenerate.

Now, consider $\phi,\phi' : U \longrightarrow \mathrm{Spec} k[x_1,\dots,x_r]$ Nisnevich coordinates near p and $\psi,\psi' : E|_u \longrightarrow \mathcal{O}_U^r$ local trivializations compatible with ϕ and ϕ' respectively. Lemma 1.31 implies that there are elements of $\mathrm{Hom}_k(\mathcal{O}_{Z,p},k)$ denoted by $\eta,\eta' : \mathcal{O}_{Z,p} \longrightarrow k$ associated to (ϕ,ψ) and (ϕ',ψ') , respectively. We then have the corresponding associated non-degenerate symmetric bilinear forms β,β' and denote by r_U, r'_U the corresponding elements of $L(U)$ associated to (ϕ,ψ) and (ϕ',ψ') respectively (see observation before 1.25). r_U and r'_U are non-vanishing by construction and so r_U/r'_U is a section in $\mathcal{O}^*(U)$. The following is a technical lemma that we will not prove. The proof is very detailed in [9] and the important thing here is that it will be useful to justify our definition of local index and its independence.

Lemma 1.33 ([9])**.** *There is an isomorphism* $\mathcal{O}_{Z,p} \longrightarrow \mathcal{O}_{Z,p}$ *given by multiplication by* r_U/r'_U *such that its pullback sends* β *to* β'.

Definition 1.34. The *local index* (or also local degree as we said before) of a global section σ of E at p is defined as the element $\mathrm{ind}_p(\sigma)$ of $\mathrm{GW}(k)$ represented by the symmetric bilinear form $\beta(x,y) = \eta(xy)$, for x, y in $\mathcal{O}_{Z,p}$.

The relevant result about the local index is a simply consequence of the technical lemmas stated before.

Theorem 1.35. *Suppose* p *is an isolated zero of a global section of* E *and that Nisnevich coordinates exist around* p. *Then the local index* $\mathrm{ind}_p(\sigma)$ *exist and is independent of the choice of:*

- *The Nisnevich coordinates* $\phi : U \longrightarrow \mathbb{A}^r$.
- *The compatible trivialization of* $E|_U$.
- $g_1, \ldots, g_r$.

Proof. Lemma 1.30 the and the correspondent isomorphism $\mathrm{Hom}_k(\mathcal{O}_{Z,p}, k) \cong \mathcal{O}_{Z,p}$ imply the existence of η and therefore $\mathrm{ind}_p(\sigma)$. Moreover, η is independent of $g_1, \ldots, g_r$ by Lemma 1.31, thus β does not depend on $g_1, \ldots, g_r$ and thus the same happens for $\mathrm{ind}_p(\sigma)$. Finally the proof of Lemma 1.33 contained in [9] shows that β is independent of the choice of Nisnevich coordinates and the theorem follows. □

Local indexes can be computed by reducing to the case where p is a k-point and using descent theory. In the case where the residue field extension $k \subseteq k(p)$ is separable, one can also change the base field for $k(p)$ and apply traces. Perhaps one of the most relevant examples in this area is the computation of the index when $X = \mathbb{A}^r$, $p = 0$ and $E = \mathcal{O}^r$ where E is given the canonical relative orientation. Kass and Wickelgren proved that in this case the Einsenbud-Khimshiashvili-Levine class of a polynomial function with an isolated zero at the origin is the local $\mathbb{A}^1$ degree [10].

Example 1.36. Let X, p and E as before. A global section σ of E can be seen as a function $\sigma : \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ and by [10] the index $\mathrm{ind}_p(\sigma)$ is the Grothendieck-Witt class of Einsenbud-Khimshiashvili-Levine signature which is equivalent to the $\mathbb{A}^1$-Brouwer degree. Moreover, if $(f_1, \ldots, f_r)$ denote the coordinate projections of σ, there are $a_{ij} \in k[x_1, \ldots, x_r]$ such that

$$f_i = \sum_{j=1}^{r} a_{ij} x_j.$$

One can choose η such that η maps the "distinguished socle" element, defined by $\det(a_{ij})$, to 1. In this case $\mathrm{ind}_p(\sigma)$ is represented by the bilinear form β in $k[x_1, \ldots, x_r]/\langle f_1, \ldots, f_r \rangle$ defined by $\beta(x,y) = \eta(xy)$.

Definition 1.37. Let $k \subseteq L$ be a separable field extension. The *Scharlau trace*, $\mathrm{Tr}_{L/k} : \mathrm{GW}(L) \longrightarrow \mathrm{GW}(k)$ is defined as the map that takes the class of a bilinear form $\beta : V \otimes V \longrightarrow L$ over L and send it to the class of β composed with the field trace, $\mathrm{tr} : L \longrightarrow k$ (where V is considered now as a k-vector space).

The good thing about the Scharlau trace is that reduces the computation of $\mathrm{ind}_p(\sigma)$ to the case where p is rational:

Theorem 1.38. *Assume $k \subseteq k(p)$ is a separable field extension where p is a isolated zero of σ such that there exists Nisnevich coordinates around p. In addition, let $X_{k(p)}$ be the base change of X to $k(p)$, $p_{k(p)}$ be the point of $X_{k(p)}$ determined by p : $Speck(p) \longrightarrow X$ and $\sigma_{k(p)}$ the base change of σ. Then*

$$ind_p(\sigma) = Tr_{k(p)/k}\, ind_{p_{k(p)}} \sigma_{k(p)}.$$

Finally we can define the Euler number associated to an algebraic vector bundle.

Definition 1.39. Let $\pi : E \longrightarrow X$ be a rank r relatively oriented vector bundle over a smooth k-scheme of dimension r. Let σ be a global section of E with isolated zeros and such that for any isolated zero there exists Nisnevich coordinates around it. The *Euler number* of E relative to σ, denoted by $e(E,\sigma)$ is defined as:

$$e(E,\sigma) := \sum_{p \in Z_0} \mathrm{ind}_p(\sigma).$$

Here Z_0 denotes the set of closed points of $Z = \{\sigma = 0\}$.

Let $\pi : E \longrightarrow X$ a vector bundle as before and let $\mathscr{E}$ be the pullback of E to $X \times \mathbb{A}^1$. $\mathscr{E}$ inherits a relative orientation from E. For a closed point $t \in \mathbb{A}^1$, $\mathscr{E}_t$ will denote the pullback of $\mathscr{E}$ to $X \otimes k(t)$ and similarly for any section s of $\mathscr{E}$, s_t will denote the pullback of s. We have the following:

Lemma 1.40 ([9])**.** *If X is a proper scheme and s is a section of $\mathscr{E}$ such that s_t has isolated zeros and Nisnevich coordinate for all closed points t of $\mathbb{A}^1$. Then, there exists a finite $\mathscr{O}(\mathbb{A}^1)$ module and a non-degenerate symmetric bilinear form β such that for any closed point t of $\mathbb{A}^1$, there is an equality $\beta_t = e(\mathscr{E}_t, s_t)$ in $k(t)$.*

Non-degenerate symmetric bilinear forms over $\mathbb{A}^1_k$ have the property that their restrictions to rational different points are stably isomorphic. This result was proved also by Kass and Wicklengren using a modified version of Harder's theorem (See [10, Lemma 31]). This implies that $e(\mathscr{E}, s_t) = e(\mathscr{E}'_t, s'_t)$ (by using 1.40 combined with the mentioned result from Kass and Wickelgren). This motivates the following:

Definition 1.41. Two sections σ and σ' of E with isolated zeros are said to be *connected by sections* with isolated zeros if there are sections s_i for $i = 0, 1, ..., N$ of $\mathscr{E}$ and rational points t_i^- and t_i^+ of $\mathbb{A}^1$ for $i = 0, 1, ..., N$ such that:

- For $i = 0, 1, .., N$ and all closed points t of $\mathbb{A}^1$, the section $(s_i)_t$ of E has isolated zeros.
- $(s_0)_{t_0^+}$ is isomorphic to σ.
- $(s_N)_{t_N^+}$ is isomorphic to σ'.
- For $i = 0, ..., N-1$, we have that $(s_i)_{t_i^+}$ is isomorphic to $(s_{i+1})_{t_{i+1}^-}$.

When the sections σ and σ' can be connected as before, i.e, by a family of sections parametrized by $\mathbb{A}^1$ with only isolated zeros, the Euler number is well defined which means that does not depend on the section σ. This is summarized in the following theorem:

Theorem 1.42. *Let $\pi : E \longrightarrow X$ be a rank r relatively oriented vector bundle on a smooth, proper scheme X of dimension r. Let σ and σ' be sections of E with isolated zeros. If after a base change by an odd degree field extension L of k, the sections can be connected by sections with isolated zeros then its Euler number coincide, namely,*

$$e(E, \sigma) = e(E, \sigma').$$

Moreover, if after the same odd degree base change of fields any two sections can be connected by sections with isolated zeros then the last equality holds for the arbitrary sections. In this case we can define the Euler number *of the vector bundle E as $e(E) := e(E, \sigma)$ for any section with isolated zeros.*

Proof. The first assertion implies the second one so it is sufficient to prove the first one. Let $k \subseteq L$ be a field extension of finite odd degree. Tensor with L provides and injective map $\mathrm{GW}(k) \longrightarrow \mathrm{GW}(L)$ so one may assume that σ and σ' can be connected by sections with isolated zeros where the scheme X is considered over k. By hypothesis σ and σ' can be connected by sections with isolated zeros, this implies that it is sufficient to prove that: For a section s of $\mathscr{E}$ such that s_t has isolated zeros for all closed points t, then $e(E, s_t) = e(E, s'_t)$ for k-rational points t and t' of $\mathbb{A}^1$. Lemma 1.40 guarantees the existence of a finite $\mathscr{O}(\mathbb{A}^1)$-module and a non-degenerate bilinear form β such that $\beta_t = e(E, s_t)$ and $\beta_{t'} = e(E, s_{t'})$. Thus the statement is reduced to proving that $\beta_t = \beta_{t'}$ in $\mathrm{GW}(k)$. This again was proved by using Harder's theorem in [10, Lemma 31]. □

1.5 Arithmetic count of lines

This section is the heart of this work and is where we will use the results from the previous sections to study the proof of the arithmetic count of lines on smooth cubic surfaces. The goal is to prove the theorem below. Before stating the theorem, is important to recall the approach of Kass and Wickelgren. The idea is to identify the arithmetic count of the lines with the Euler number of a vector bundle on the Grassmannian $\mathrm{Gr}(4,2)$ of lines in $\mathbb{P}^3$. So the general idea behind is something similar to the one we already saw in the Motivation 1.1, in particular in

the proof of Theorem 1.1. Behind the scenes is acting some kind of "categorization" where the arithmetic is replaced by appropriate classes using Euler numbers. This is important to highlight and it is perhaps one of the main reasons that have motivated the author to study these topics. The works of Morel and Voevodsky have opened the door to new strategies and tools to prove arithmetical results and many others. To mention one of the most relevant, we have the proof of the Milnor conjecture made by Voevodsky. One good reference regarding the applications in Enumerative Geometry is Levine's article "Toward an enumerative geometry with quadratic forms" [12].
The main theorem of this section is the following:

Theorem 1.43. *Let k be a field and let $X \subseteq \mathbb{P}^3_k$ be a smooth cubic surface. Then the lines on X satisfy:*

$$\sum_{\text{lines } l \text{ with field } L} \Big((\textit{hyperbolic lines}) Tr_{L/k}(\langle 1 \rangle) + \sum_{h \in L^*/(L^*)^2} (\textit{elliptic lines of type } h) \cdot Tr_{L/k}(\langle h \rangle) \Big)$$

$$= 15\langle 1 \rangle + 12\langle -1 \rangle.$$

1.5.1 The distinguished bundle and its orientation

Let x_1, x_2, x_3, x_4 denote the dual basis of the standard basis of k^4 and let $B = \{e_1, e_2, e_3, e_4\}$ be other basis for k^4 with associated dual basis $\{\phi_1, \phi_2, \phi_3, \phi_4\}$. We start by giving an orientation to the vector bundle on which we will apply all the theory studied before.

Definition 1.44. Let $\mathscr{S}$ the universal subbundle on $\mathrm{Gr}(4,2)$ and $\mathscr{Q}$ the quotient bundle on the same Grassmannian. Set $\mathscr{E} := \mathrm{Sym}^3(\mathscr{S}^*)$.
The global section σ_f associated to an homogeneous degree 3 polynomial $f \in \mathrm{Sym}^3((k^4)^*)$ is defined as the image of f under the homomorphism $\mathrm{Sym}^3(\mathscr{O}^4) \longrightarrow \mathscr{E}$ induced by the inclusion $\mathscr{S} \hookrightarrow \mathscr{O}^4$.

As we saw in the first sections, if $S \subseteq k^4$ is a 2-dimensional subspace, then the fiber of $\mathscr{E}$ at the corresponding point of $\mathrm{Gr}(4,2)$ can be think as the space of homogeneous degree 3 polynomials on S, namely $\mathrm{Sym}^3(S^*)$. σ_f evaluated at this point is just the restriction $f|_S$.

The tangent bundle to $\mathrm{Gr}(4,2)$ admits a description in the language of universal bundles (see [4, Section 3.2.4]) as follows:

$$T(\mathrm{Gr}(4,2)) = \mathscr{H}om_{\mathrm{Gr}(4,2)}(\mathscr{S}, \mathscr{Q}) = \mathscr{S}^* \otimes \mathscr{Q}.$$

In the following we will focus on give an explicit relative orientation of $\mathscr{E}$. We will need the following:

Definition 1.45. Consider the basis B defined before and define:

$$\begin{aligned}\tilde{e}_1 &:= e_1\\ \tilde{e}_2 &:= e_2\\ \tilde{e}_3 &:= x e_1 + y e_2 + e_3\\ \tilde{e}_4 &:= x' e_1 + y' e_2 + e_4\end{aligned}$$

The elements we just defined $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4$ are in $\left(k[x, x', y, y']\right)^4$ and a computation show that they form a basis. We will denote its dual basis by $\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4$.

Proposition 1.46. *The morphism* $Spec\left(k[x, x', y, y']\right) = \mathbb{A}^4_k \longrightarrow Gr(4,2)$ *is an open immersion. This morphism has the property that the pullback of* $\mathcal{S}$ *under this morphism gives the subspace:*

$$k[x, x', y, y']\tilde{e}_3 + k[x, x', y, y']\tilde{e}_4 \subseteq \left(k[x, x', y, y']\right)^4.$$

Proof. Denote $\mathrm{Gr}(4,2) = G$. If $q : \mathcal{O}^4_G \longrightarrow \mathcal{Q}$ and define a morphism $\mathcal{O}^2_G \longrightarrow \mathcal{O}^4_G$ by $(a, b) \mapsto a e_1 + b e_2$. Therefore if $q : \mathcal{O}^4_G \longrightarrow \mathcal{Q}$ is surjective, we can consider the composition:

$$\mathcal{O}^2_G \longrightarrow \mathcal{O}^4_G \xrightarrow{q} \mathcal{Q}$$

By [6, Lemma 9.7.4.6] the subfunctor of G parametrizing quotients q such that the last composition is an isomorphism, is represented by the morphism $\mathbb{A}^4_k = \mathrm{Spec}(k[x, x', y, y']) \longrightarrow G$. The subfunctor is open and therefore the morphism is an open immersion. □

We denote by $U(B) \subseteq G$ the image of the morphism $\mathrm{Spec}\left(k[x, x', y, y']\right) = \mathbb{A}^4_k \longrightarrow G$. The collection $\{U(B)\}$ is an standard open cover that trivializes the universal subbundle and the quotient bundle.

Lemma 1.47. *In terms of the last collection* $\{U(B)\}$, *the restrictions* $TG|_{U(B)}$ *and* $\mathcal{E}|_{U(B)}$ *have bases given by:*

$$\beta_0 := \{\tilde{\phi}_3 \otimes \tilde{e}_1, \tilde{\phi}_4 \otimes \tilde{e}_1, \tilde{\phi}_3 \otimes \tilde{e}_2, \tilde{\phi}_4 \otimes \tilde{e}_2\} \tag{1-5}$$

$$\beta_1 := \{\tilde{\phi}_3, \tilde{\phi}_3^2\tilde{\phi}_4, \tilde{\phi}_3\tilde{\phi}_4^2, \tilde{\phi}_4^3\} \tag{1-6}$$

respectively. In particular, $\mathcal{H}om(\Lambda^4 TG, \Lambda^4\mathcal{E})|_{U(B)}$ *is freely generated by a section* $\nu(B)$ *that maps the wedge products of the sections in* β_0 *to the wegde product of the sections in* β_1.
If a second base is given, say B', *and is such that* $span(e_3, e_4) = span(e'_3, e'_4)$, *the sections* $\nu(B)$ *and* $\nu(B')$ *satisfy:*

$$\nu(B') = \left(\frac{1}{(ad - bc)^2(\alpha\delta - \beta\gamma)}\right)^2 \nu(B).$$

Where $\alpha, \beta, \gamma, \delta, a, b, c, d$ are defined by the following:

$$\tilde{e}'_1 = \alpha \tilde{e}_1 + \beta \tilde{e}_2 + z$$
$$\tilde{e}'_2 = \gamma \tilde{e}_1 + \delta \tilde{e}_2 + w$$
$$\tilde{e}'_3 = a \tilde{e}_3 + b \tilde{e}_4$$
$$\tilde{e}'_4 = c \tilde{e}_3 + d \tilde{e}_4$$

here $z, w \in span(\tilde{e}_3, \tilde{e}_4)$.

Proof. The fact β_0 and β_1 are bases is clear from the definitions and clearly implies that $\nu(B)$ maps the elements of β_0 to the elements of β_1. Now, denote by β'_0 and β'_1 the corresponding basis as in (1-5) and (1-6) associated to the basis B'. Let $\det_0$ be the determinant of the change of base matrix related to β_0 and β'_0 and $\det_1$ be the determinant of the change of base matrix related to β_1 and β'_1. Since $\nu(B)$ maps the wedge products of β_0 to the wedge products of the elements of β_1 and $\nu(B')$ does the same β'_0 and β'_1, a computation on the respective bases shows that:

$$\nu(B') = \frac{\det_1}{\det_0} \nu(B).$$

By defining:

$$\tilde{e}'_3 = a \tilde{e}_3 + b \tilde{e}_4$$
$$\tilde{e}'_4 = c \tilde{e}_3 + d \tilde{e}_4$$

we have that the elements of the dual basis are related by:

$$\tilde{\phi}'_3 = A \tilde{\phi}_3 + C \tilde{\phi}_4 + Z$$
$$\tilde{\phi}'_4 = B \tilde{\phi}_3 + D \tilde{\phi}_4 + W$$

where $Z, W \in \text{span}(\tilde{\phi}_1, \tilde{\phi}_2)$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The matrix with uppercase letters is the change of base matrix that relate $\tilde{\phi}_3, \tilde{\phi}_4$ with $\tilde{\phi}'_3, \tilde{\phi}'_4$ thanks to the fact that $\tilde{\phi}_1, \tilde{\phi}_2$ are zero on $\mathscr{S}|_{U(B)}$. Therefore, using the expressions for $\tilde{e}'_1$ and $\tilde{e}'_2$ it can be shown that

$$\det_0 = \frac{(\alpha\delta - \beta\gamma)^2}{(ad - bc)}$$
$$\det_1 = \frac{1}{(ad - bc)^6}$$

and the result follows. □

Define $\mathscr{L}$ to be the product $\Lambda^2\mathscr{Q}^*\otimes\Lambda^2\mathscr{S}^*\otimes\Lambda^2\mathscr{S}$. By taking exterior powers in the sequence $0\longrightarrow\mathscr{S}\longrightarrow\mathscr{O}_G^4\longrightarrow\mathscr{Q}\longrightarrow 0$, it turns out that $\mathscr{L}$ is isomorphic to $\Lambda^2\mathscr{S}^2$. The tensor product of $\mathscr{L}$ with itself will define the required relative orientation for $\mathscr{E}$, the next corollary justify this and is basically an application of Lemma 1.47.

Corollary 1.48. *There is an isomorphism $j:\mathscr{H}om(\Lambda^4TG,\Lambda^4\mathscr{E})\longrightarrow\mathscr{L}\otimes\mathscr{L}$ such that the restriction to $U(B)$ maps $\nu(B)$ to $(\tilde{\phi}_1\wedge\tilde{\phi}_2)\otimes(\tilde{\phi}_3\wedge\tilde{\phi}_4)\otimes(\tilde{\phi}_3\wedge\tilde{\phi}_4)$. Moreover, this isomorphism is unique.*

As we said before, the line bundle $\mathscr{L}$ and the isomorphism j provide the relative orientation we wanted:

Definition 1.49. The *distinguished relative orientation* of $\mathscr{E}$ is the pair $(\mathscr{L},j)$.

The relative orientation $(\mathscr{L},j)$ is determined in a unique way thanks to the fact that the line bundle $\mathscr{L}$ is a square root of $\mathscr{H}om(T(G),\mathscr{E})$ (The Picard group of G has no torsion) so there is no other choice of line bundle. If we consider the morphisms $a\cdot j$for $a\in k$, these are alsp isomorphism between $\mathscr{H}om(\Lambda^4TG,\Lambda^4\mathscr{E})$ and $\mathscr{L}\otimes\mathscr{L}$, however j has the property that the local index of σ_f at a zero is equal to the type of the corresponding associated line. Moreover j is defined over $\mathbb{Z}$ and therefore is characterized in a unique way. Let's see how we can identify the local index of σ_f at a zero with the type of the corresponding line.

Lemma 1.50 ([9])**.** *Let $S\subseteq k^4$ be a subspace such that $f|_S$ vanishes. Then the differential of σ_f at the corresponding k-point of G is given by the map $S^*\otimes\mathscr{Q}\longrightarrow Sym^3(S^*)$ that sends $\phi\otimes(v+S)\mapsto\left(\frac{\partial f}{\partial v}\big|_S\cdot\phi\right)$.*

By construction, the zero locus of σ_f is the set of lines contained in the cubic surface defined as the zero set of f.

Lemma 1.51. *Let $S=k\cdot e_1+k\cdot e_2\subseteq k^4$ be a subspace. Then the derivative of σ_f at a zero defined by S corresponds to*

$$Res\left(\frac{\partial f}{\partial e_1}(xe_3+ye_4),\frac{\partial f}{\partial e_2}(xe_3+ye_4)\right)$$

where $e_3,e_4\in k^4$ are such that e_1,e_2,e_3,e_4 is a basis for k^4. The resultant is seen in $k/(k^*)^2$.

Proof. One can compute the matrix of the differential $\mathrm{Hom}(S,\mathscr{Q})\longrightarrow\mathrm{Sym}^3(S^*)$ with respect to the bases $\phi_3\otimes e_1,\phi_4\otimes e_1,\phi_3\otimes e_2,\phi_4\otimes e_2$ and $\phi_3^3,\phi_3^2\phi_4,\phi_3\phi_4^2,\phi_4^3$ (see 1.45 and 1.50). Such matrix is given by:

$$A=\begin{pmatrix} a_{1,0,2,0} & 0 & a_{0,1,2,0}, & 0\\ a_{1,0,1,1} & a_{1,0,2,0} & a_{0,1,1,1} & a_{0,1,2,0}\\ a_{1,0,0,2} & a_{1,0,1,1} & a_{0,1,0,2} & a_{0,1,1,1}\\ 0 & a_{1,0,0,2} & 0 & a_{0,1,0,2}\end{pmatrix}$$

The last matrix is also the Sylvester matrix of the polynomials $\frac{\partial f}{\partial e_1}(xe_3+ye_4)$ and $\frac{\partial f}{\partial e_2}(xe_3+ye_4)$, where they are considered as polynomials in ϕ_3 and ϕ_4. By definition of the determinant of a

Sylvester matrix, such determinant is the resultant of the last two polynomials. On the other hand, by Lemma 1.50 and the definition of the distinguished orientation, the differential of σ_f is the class of det(A) and the lemma follows. □

By the Lemma 1.18 we know that the type of a line equals the class of the resultant of the partial derivatives of f with respect to e_1 and e_2 and the last lemma says that such resultant is equal to the derivative of σ_f at a zero. This combination of the two results imply the following:

Corollary 1.52. *The type of a line l on a smooth cubic surface defined by $X = \{f = 0\}$ is equal to the index of σ_f at the corresponding zero. In particular the line is hyperbolic if and only if the index is $\langle 1\rangle$.*

Corollary 1.53. *The section σ_f where f defines a smooth cubic surface has only simple zeros. Moreover, if the surface is not smooth, i.e, is possibly singular and is defined by $\{f = 0\}$, a line on the surface corresponds to a simple zero of σ_f.*

Proof. First start by extending scalars to the algebraic closure of k, $\bar{k}$. With the same notations as in Lemma 1.51 is sufficient to show that the resultant

$$\mathrm{Res}\left(\frac{\partial f}{\partial e_1}(x e_3 + y e_4), \frac{\partial f}{\partial e_2}(x e_3 + y e_4)\right)$$

is nonzero (The resultant of the partial derivatives is the discriminant, and if vanishes at a point, then the zero is multiple). If its zero, there will be a vector $v \in k^4 \smallsetminus \{0\}$ such that

$$\frac{\partial f}{\partial e_1}(v) = \frac{\partial f}{\partial e_2}(v) = 0.$$

Since f vanishes at $S = ke_3 + ke_4$, we also have that $\frac{\partial f}{\partial e_3}(v) = \frac{\partial f}{\partial e_4}(v) = 0$ and so the subspace $k \cdot v \subseteq k^4$ defines a point in $\mathbb{P}^3_k$ contained in the singular locus of X. However X is smooth along the line so we get a contradiction. □

Finally we notice that in this particular case, the existence of Nisnevich coordinates is satisfied because the field of a definition of a line is always a separable extension of k (See 1.23).

Corollary 1.54. *Let l be a line contained in a smooth cubic surface. Then the field of definition of l is a separable extension of the ground field k.*

Proof. Recall that for schemes defined over a field k, being geometrically reduced is equivalent to have separable extensions $k(p)$ of k for any closed point. Therefore, if L denotes the field of definition of the line, the natural inclusion $\mathrm{Spec}(L) \longrightarrow \{\sigma_f = 0\}$ defines a connected component of $\{sigma_f = 0\}$. Corollary 1.53 implies that $\{\sigma_f = 0\}$ is geometrically reduced, since connected components of a geometrically reduced scheme are also geometrically reduced it follows that Spec(L) is itself geometrically reduced. By the equivalence it follows that L/k is separable. □

1.5.2 The Euler number is well defined

Now we will show that in this particular case, the sections of the distinguished vector bundle can be connected by sections with isolated zeros or in other terms, we need to show that there are many sections of $\mathscr{E}$ that in some sense, avoid the locus of sections with non isolated zeros. This will imply the good definition of the Euler number $e(\mathscr{E})$ (see 1.42).

Definition 1.55. Consider the vector space $(k^{19})^*$ and denote the dual of its standard basis by $\{a_{i,j,k,l} : i+j+k+l=3\}$. Define:

$$V := \left\{\sum a_{i,j,k,l}\, x_1^i x_2^j x_3^k x_4^l = 0\right\} \subseteq \mathbb{P}^{19}_k \times_k \mathbb{P}^3_k.$$

and denote by $V_{\text{sing}} \subseteq V$ the non smooth locus of $V \longrightarrow \mathbb{P}^{19}_k$, $I_l \subseteq V$ the intersection $V_{\text{sing}} \cap \{\text{Hess}(f)=0\}$ and I_2 the closure of the complement of the diagonal in $V_{\text{sing}} \times_{\mathbb{P}^{19}_k} V_{\text{sing}}$.

Lemma 1.56. *The images under the respective projections of I_1 and I_2 onto $\mathbb{P}^19_k$ are closed subsets of dimensions 17.*

Proof. As we did for the Corollary 1.53, is sufficient to prove the assertion after extending scalars to the algebraic closure of k, so without loss of generality let's assume $k = \bar{k}$. Let $\pi_1 : I_1 \longrightarrow \mathbb{P}^3_k$ and $\pi_2 : I_2 \longrightarrow \mathbb{P}^3 \times_k \mathbb{P}^3_k$ be the respective projections. The fiber of the subspace $(0,0,0,1)\cdot k$ under the projection is defined by equations:

$$a_{1,0,0,2} = a_{0,1,0,2} = a_{0,0,1,2} = a_{0,0,0,3}$$

and

$$a_{2,0,0,1}a^2_{0,1,1,1} - a_{1,0,1,1}a_{1,1,0,1}a_{0,1,1,1} + a_{0,2,0,1}a^2_{1,0,1,1} + a_{0,0,2,1}a^2_{1,1,0,1} - 4a_{0,0,2,1}a_{0,2,0,1}a_{2,0,0,1} = 0.$$

These equations form a regular sequence and therefore they define an irreducible subvariety of $\mathbb{P}^19_k$ of dimension $14 = 19-5$. Same thing is true for all other fibers of π_1 so I_1 is irreducible of dimension $14+3=17$. Clearly this implies I_1 is closed. An analogous argument proves the same for I_2. □

Definition 1.57. Define $\mathscr{D}_0 \hookrightarrow H^0(G,\mathscr{E})$ to be the k-points of the affine cone over the union $\pi_1(I_1)$ and $\pi_2(I_2)$. That is, $\mathscr{D}_0$ si the subset of f's such that the variety $\{f=0\}\otimes \bar{k}$ has at least two singularities or has a singularity at which $\text{Hess}(f)=0$.

It turns out that $\mathscr{D}_0$ contains the global sections with a non isolated zero:

Lemma 1.58 ([9])**.** *Let $f \in H^0(G,\mathscr{E}) \smallsetminus \mathscr{D}_0$. Then the associated section σ_f has only isolated zeros.*

Theorem 1.59. *If σ and σ' are sections of $\mathscr{E} = Sym(\mathscr{S}^*)$ with isolated zeros, then they can be connected by sections with isolated zeros.*

Proof. It is enough to prove that after extending scalars to an odd degree field extension, any two sections of $H^0(G,\mathscr{E})\setminus\mathscr{D}_0$ can be connected by affine lines that not intersect $\mathscr{D}_0$ (see **??** and 1.58). This is thanks to the fact that $\mathscr{D}_0$ equals the k-points of a subvariety of codimension at least 2. Indeed, if $f,g\in H^0(G,\mathscr{E})$ and k is finite, then after a odd degree extension one can choose a 3-dimensional subspace $S\subseteq H^0(G,\mathscr{E})$ such that $S\cap\mathscr{D}_0$ equals the k-points of the cone over a 0-dimensional subscheme. $\mathscr{D}_0$ is then a finite union of 1-dimensional subspaces. By choosing a larger extension of odd degree the number of 1 dimensional subspaces contained in $\mathscr{D}_0$ is less than the number of 2-dimensional subspaces of S that contain f. Therefore, choose T_f and T_g subspaces of S of dimension 2 and such that they are not included $\mathscr{D}_0$ but they contain f and g, respectively. Counts of dimensions show that $T_f\cap T_g\neq 0$ so there's an element $h\in T_f\cap T_g\setminus\{0\}$. The construction of S and $\mathscr{D}_0$ allow to conclude the following: The line joining f and h and the line joining h and g are both disjoint from $\mathscr{D}_0$. So the claim is proved and the Theorem follows. □

1.5.3 The main result

We start by recalling the main theorem we stated before:

Theorem 1.60. *Let k be a field and let $X\subseteq\mathbb{P}^3_k$ be a smooth cubic surface. Then the lines on X satisfy:*

$$\sum_{\text{lines } l \text{ with field } L}((\textit{hyperbolic lines})Tr_{L/k}(\langle 1\rangle) + \sum_{h\in L^*/(L^*)^2}(\textit{elliptic lines of type } h)\cdot Tr_{L/k}(\langle h\rangle) = 15\langle 1\rangle + 12\langle -1\rangle. \tag{1-7}$$

Proof. The Euler number of the vector bundle $\mathscr{E}$ is well defined thanks to 1.59. Furthermore 1.18 combined with 1.51 show that the left hand side of (2-7) is identified with the Euler number $e(\mathscr{E})$ and hence it is independent of the choice of the surface. Then, the proof can be done by computing the euler number for two particular simple surfaces.

First, let X be the cubic surface defined by $f=x_1^3+x_2^3+x_3^3+x_4^3$ over a field k with $\mathrm{char}(k)\neq 3$ that not contains a primitive third root of unity ω_3. This surface contains 3 lines with field of definition k and 12 with field of definition $L=k(\omega_3)$ given by the following: consider the subspace

$$(-1,\omega_3^i,0,0)k(\omega_3)+(0,0,-1,\omega_3^j)k(\omega_3)\subseteq L^4 \tag{1-8}$$

where $i,j=0,1,2$. This subspace defines a morphism Spec(L) $\longrightarrow G$ where the image is a line contained in X. Allowing permutations on the coordinates of k^4 one gets another two

morphism of this kind. There are 9 choices for i, j and hence we obtain 27 morphisms of the form $\mathrm{Spec}(L) \longrightarrow G$. Its images are the lines we wanted. The signed count with weights equal to the degree of the field of definition is also 27 so there are no more lines other than this 27.For another approach of this see 1.2.
Notice that a line defined by (1-8) satisfies that $\mathrm{Res}\left(\frac{\partial f}{\partial e_1}\Big|_S, \frac{\partial f}{\partial e_2}\Big|_S\right) = 9$, where $e_1 = (1,0,0,0)$ and $e_2 = (0,0,1,0)$, thus according to Proposition 1.18, such a line is hyperbolic. All the other lines can be obtained by automorphisms, thus all lines are hyperbolic and:

$$e(\mathscr{E}) = 3\langle 1\rangle + 12 \cdot \mathrm{Tr}_{k(\omega_3)/k}(\langle 1\rangle) = 3\langle 1\rangle + 12 \cdot (\langle 2\rangle + \langle 2(-3)\rangle)$$

When k contains a primitive third root of the unity one can use the same argument and now the lines are in X considered over k, thus $e(\mathscr{E}) = 27\langle 1\rangle$.

Now, if $\mathrm{char}(k) \neq 5$ let's consider the smooth cubic surface defined by

$$f = \sum_{i,j=1, i\neq j}^{4} x_i^2 x_j + 2\sum_{i=1}^{4} x_1 x_2 x_3 x_i^{-1}.$$

Notice that in $\mathrm{char}(k) \neq 3$, f can be also written as $f = \sum_{i,j=1,i\neq j}^{4} x_i^2 x_j + 2\sum_{i=1}^{4} x_1 x_2 x_3 x_i^{-1}$. In this case, the main tool to analyze the lines on X focuses on action of the symmetric group S_5 on X which is defined as follows: The action of the group S_5 is defined as follows:

$$\sigma(x_i) = \begin{cases} -x_1 - x_2 - x_3 - x_4 & \text{if } \sigma(i) = 5 \\ \\ x_{\sigma(i)} & \text{otherwise} \end{cases}$$

for $\sigma \in S_5$. The polynomial f is invariant under this equation and so induces an action on X. When the field k does not contain $\sqrt{5}$, there are lines on X defined by:

$$(1,-1,0,0)k + (0,0,1,-1)k \subseteq k^4 \tag{1-9}$$

$$(2,\alpha,\hat{\alpha},\hat{\alpha})k(\sqrt{5}) + (\alpha,\hat{\alpha},\hat{\alpha},\hat{\alpha})k(\sqrt{5}) \subseteq k(\sqrt{5})^4 \tag{1-10}$$

where $\alpha = \frac{-1+\sqrt{5}}{2}$ and $\hat{\alpha} = \frac{-1-\sqrt{5}}{2}$. The respective fields of definition of the lines are equal to k and $k(\sqrt{5})$. The type of the lines can be computed using the same techniques as in the first case, namely, for (1-9) using the partial derivatives with respect to $(1,0,0,0)$ and $(0,0,1,0)$ one can see that the line is hyperbolic. For (1-10) computing with respect to $(0,1,0,0)$ and $(0,0,0,1)$, its type is $\frac{-25}{2}(5+\sqrt{(5)})$.

The action of S_5 provides orbits for the lines. For the first one, its orbit has 15 elements and for the second one 6 elements. In addition, the type is invariant under automorphisms and therefore:

$$e(\mathscr{E}) = 15\langle 1\rangle + 6\mathrm{Tr}_{k(\sqrt{5})/k}\left\langle \frac{-25}{2}(5+\sqrt{5})\right\rangle = 15\langle 1\rangle + 12\langle -5\rangle.$$

If k contains $\sqrt{5}$, the previous argument also holds except for the fact that the 6 lines on with field $k(\sqrt{(5)})$ become 12 lines defined over k: half of them have type equal to $\frac{-(5+\sqrt{5})}{2}$ and the other half $\frac{-(-5-\sqrt{5})}{2}$. Therefore:

$$e(\mathscr{E}) = 15\langle 1\rangle + 6\left\langle -\frac{(5+\sqrt{5})}{2}\right\rangle + 6\left\langle \frac{-(5-\sqrt{5})}{2}\right\rangle$$

By reducing to the case where the field is either $\mathbb{F}_p$ or $\mathbb{Q}$, it can be shown that all the classes computed equal $15\langle 1\rangle + 12\langle -1\rangle$ (see [9]). □

As an application of the main theorem and a result in field theory [9, Lemma 58] one has the following consequence:

Corollary 1.61. *If $X \subseteq \mathbb{P}^3_{\mathbb{F}_q}$ is a smooth cubic surface. Then the number of elliptic lines on X with field of definition $\mathbb{F}_{q^\alpha}$ for α odd plus the number of hyperbolic lines on X with field of definition $\mathbb{F}_{q^\beta}$ with β even is congruent with 0 modulo 2.*

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