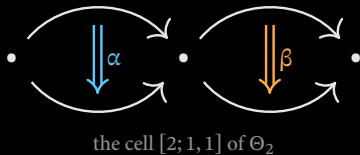


A homotopy coherent cellular nerve for bicategories, I

After A. Campbell, *Adv. Math.* 368 (2020), §§1–5

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Working group, Grenoble 2026–2027 · Talk 4



This talk:

1. Bicategories and simplices (§§1--3) Simplicial sets cannot host non-invertible 2-cells as $(\infty, 1)$ -data: the Duskin nerve is **fully faithful**, but it is a quasi-category **only when all 2-cells are invertible**.
2. Change the shapes (§4) Replace Δ by Θ_2 and simplices by **normal pseudofunctors** $[n; m] \rightarrow B$: a **fully faithful** nerve, determined by its cells of degree ≤ 3 .
3. Homotopy theory (§§5--6) This nerve is **right Quillen** from Lack's model structure to Ara's, with fully faithful derived functor; nerves of bicategories are **2-quasi-categories**.

§ 1

Motivation, state of the art, and statements

1. **Motivation, state of the art, and statements**
2. Recollections on bicategories
3. The Duskin nerve
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7. Summary and outlook

The one-dimensional template

- ▶ Grothendieck's nerve $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$, $N(C)_n = \mathbf{Cat}([n], C)$, is **fully faithful**; its image is the simplicial sets with **unique** inner horn fillers (Lurie, HTT, Prop. 1.1.2.2).
- ▶ Drop uniqueness: **quasi-categories** (Joyal 2002, Def. 1.1; Lurie, HTT, Def. 1.1.2.4). So $N : \mathbf{Cat} \hookrightarrow \mathbf{qCat}$: *quasi-category theory generalises category theory*.
- ▶ Homotopically: $\tau_1 \dashv N$ is Quillen (canonical model structure $\leftrightarrow$ Joyal's), and the counit $\tau_1 NC \cong C$ is invertible: the homotopy theory of categories **embeds** into that of $(\infty, 1)$ -categories.

Campbell's paper is the two-dimensional version of this picture for Ara's Θ_2 -model of $(\infty, 2)$ -categories (left adjoints on top):

$$\mathbf{Cat} \begin{array}{c} \xleftarrow{\tau_1} \\ \perp \\ \xrightarrow{N} \end{array} \mathbf{sSet} \quad \rightsquigarrow \quad \mathbf{Bicat}_s \begin{array}{c} \xleftarrow{\tau_b} \\ \perp \\ \xrightarrow{N} \end{array} [\Theta_2^{\text{op}}, \mathbf{Set}]$$

Nerves of bicategories: state of the art

Bicategories (Bénabou 1967, (1.1)) are the weak 2-categories met in practice; $(\infty, 2)$ -categories have many models, all equivalent (Bergner–Rezk 2013, 2020; Barwick–Schommer-Pries 2021, Thm. 1.1; Gagna–Harpaz–Lanari 2022, Thm. 7.9). For each model one wants a nerve that is **fully faithful** and lands in **fibrant objects**:

model of $(\infty, 2)$ -categories	nerve	references
qCat -enriched categories (saturated) 2-complicial sets	change of base N_* (2-categories) Roberts–Street–Duskin nerve, equivalences marked	Riehl–Verity 2022, Digression 1.4.2 Street 1987, §5; Duskin 2002, §6; Verity 2008, Thm. 266; Gurski 2009; Ozornova–Rovelli 2021a, Thm. 5.2
qCat -enriched Segal categories ∞ -bicategories	Lack–Paoli 2-nerve, then change of base scaled Duskin nerve	Tamsamani 1999; Lack–Paoli 2008, Thm. 3.7; Simpson 2012, Thm. 21.2.1 Gagna–Harpaz–Lanari 2022, Def. 8.1; Lurie 2009, Def. 4.2.8
Θ_2-sets (2-quasi-cat.)	strict N_s fails ; coherent N	Berger 2002, Thm. 1.12; Ara 2014, Cor. 7.11; Leinster 2002, Def. J; Campbell 2020
Rezk Θ_2 -spaces	$t^! \circ N$	Rezk 2010, §11; Campbell 2020, Thm. 8.12 (Talk 5)

Afterwards: all these nerves agree modulo change of model (Moser–Ozornova–Rovelli 2022, Thms. A and B).

The obstruction: the strict cellular nerve

- Talk 3: the **strict nerve** $N_s : 2\text{-Cat} \rightarrow [\Theta_2^{\text{op}}, \text{Set}]$, singular functor of $\Theta_2 \hookrightarrow 2\text{-Cat}$, is **fully faithful** (Berger 2002, Thm. 1.12). So full faithfulness is *not* the issue. Fibrancy is:

Theorem (Ara 2014, Prop. 7.10 and Cor. 7.11) For a 2-category A , $N_s A$ is a 2-quasi-category *iff* A is **rigid** (no non-identity invertible 2-cell). In particular $J_2 = N_s(\mathbb{I}_2)$ is **not** a 2-quasi-category.

- Slogan of quasi-category theory: a map $A \rightarrow X$ into a 2-quasi-category is, **up to homotopy**, a “weak functor $A \rightarrow X$ ”.

Why this is to be expected

- By full faithfulness, maps $N_s C \rightarrow N_s D$ correspond only to *strict* 2-functors $C \rightarrow D$ (Campbell 2020, Rem. 3.3). Alone this is no contradiction: for **rigid** D , normal pseudofunctors into D are strict (§4) and $N_s D$ is fibrant.

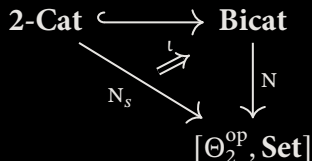
Lack's 2-category A (Lack 2002, Ex. 3.1) One object; underlying category the group $\mathbb{Z}$; a unique 2-cell $m \Rightarrow n$ exactly when $m - n$ is even (so all 2-cells are invertible and A is **not rigid**). The 2-functor $A \rightarrow \mathbb{Z}/2$, $n \mapsto n \bmod 2$, is a **biequivalence**; but a 2-functor $\mathbb{Z}/2 \rightarrow A$ sends the generator to some n with $2n = 0$: it is **trivial**.

- The real obstruction (Ara 2014, §7): if $N_s A$ were fibrant, the weak equivalence $N_s(A \rightarrow \mathbb{Z}/2)$ would have a homotopy inverse $N_s(\mathbb{Z}/2) \rightarrow N_s A$, i.e. a 2-functor $\mathbb{Z}/2 \rightarrow A$; it is trivial, so it cannot be one. (N_s sends biequivalences to weak equivalences: Campbell 2020, Thm. 10.11, Talk 5.)
- Taking the slogan literally, with weak functor = **normal pseudofunctor**, is the definition of the coherent nerve (Campbell 2020, Rem. 3.3).

Campbell's answer and today's statements

Definition (Leinster 2002, Def. J; Campbell 2020, Def. 3.5) The **coherent nerve** $N : \mathbf{Bicat} \rightarrow [\Theta_2^{\text{op}}, \mathbf{Set}]$ is

$$(NB)_{n(m)} = \{\text{normal pseudofunctors } [n; m] \longrightarrow B\}.$$



ι_A invertible iff A rigid; always a weak equivalence (Theorem C, Talk 5).

Today's theorems (Campbell 2020)

- ▶ **Thm. 3.18:** $N : \mathbf{Bicat} \rightarrow [\Theta_2^{\text{op}}, \mathbf{Set}]$ is **fully faithful**.
- ▶ **Thm. 5.10:** on bicategories and strict morphisms, $\tau_b \dashv N : \mathbf{Bicat}_s \rightleftarrows [\Theta_2^{\text{op}}, \mathbf{Set}]$ is a Quillen adjunction (Lack $\leftrightarrow$ Ara), and RN is **fully faithful**.
- ▶ **Theorem A** (Cors. 5.11, 5.13): every NB is a **2-quasi-category**; $N : \mathbf{Bicat} \rightarrow 2\text{-qCat}$ is fully faithful.

§ 2

Recollections on bicategories

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Bicategories

Definition (Bénabou 1967, (1.1); Johnson--Yau 2021, Def. 2.1.3) A **bicategory** B : objects $a, b, \dots$; **hom-categories** $B(a, b)$, with objects the **1-cells** $f : a \rightarrow b$ and morphisms the **2-cells** $\alpha : f \Rightarrow g$ (vertical composition $\beta \cdot \alpha$); identities $\mathbb{1}_a$; composition functors $B(b, c) \times B(a, b) \rightarrow B(a, c)$ (horizontal composition $\beta * \alpha$, whiskerings $h\alpha, \beta f$); invertible **associator** and **unitors**

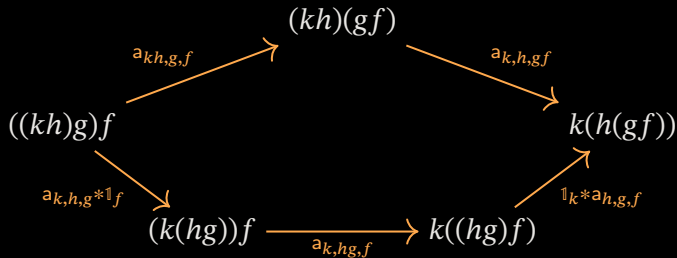
$$a_{h,g,f} : (hg)f \cong h(gf), \quad l_f : \mathbb{1}_b f \cong f, \quad r_f : f \mathbb{1}_a \cong f,$$

satisfying the pentagon and triangle axioms. A **2-category**: a, l, r are identities.

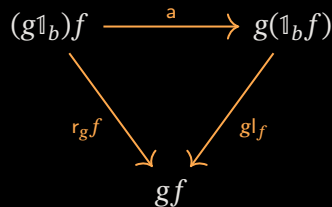
Examples (Bénabou 1967, (2.2), (2.6), (2.5); Johnson--Yau 2021, Exs. 2.1.22, 2.1.26)

- ▶ 2-categories: **Cat**; every $[n; m]$; the free-living isomorphism $\mathbb{1}$ and invertible 2-cell $\mathbb{1}_2$.
- ▶ Monoidal categories $(V, \otimes, \mathbb{1})$ = one-object bicategories ΣV .
- ▶ Genuinely weak: $\text{Span}(C)$, Bimod (composition by a universal property). Categories = locally discrete bicategories.

The coherence axioms



pentagon: the two rebracketings of $((kh)g)f$ agree



triangle: a is compatible with the unitors

By coherence, every diagram of constraints commutes; we never need more than these two.

Morphisms: normal pseudofunctors

Definition (Johnson--Yau 2021, Def. 4.1.2; Campbell 2020, §3.4) A **lax functor** $F : A \rightarrow B$: a map on objects, functors $F : A(a, b) \rightarrow B(Fa, Fb)$, and 2-cells

$$\varphi_{g,f} : FgFf \Rightarrow F(gf) \text{ (natural),} \quad \varphi_a : \mathbb{1}_{Fa} \Rightarrow F(\mathbb{1}_a),$$

satisfying an associativity axiom (a hexagon involving a) and two unit axioms. It is **pseudo** if the φ 's are invertible, **normal** if $F(\mathbb{1}_a) = \mathbb{1}_{Fa}$ and $\varphi_a = \mathbb{1}$, **strict** if all φ 's are identities.

$$\underbrace{\mathbf{2-Cat}}_{\text{2-functors}} \subset \underbrace{\mathbf{Bicat}_s}_{\text{strict morphisms}} \subset \underbrace{\mathbf{Bicat}}_{\text{normal pseudofunctors}}$$

Why normal? Nerves have **degeneracies**: the degenerate 1-simplex on a is $\mathbb{1}_a$, and maps of nerves preserve degeneracies, forcing $F(\mathbb{1}_a) = \mathbb{1}_{Fa}$. Nothing is lost up to equivalence (Lack–Paoli 2008, Introduction). For normal F the only genuine data are the $\varphi_{g,f}$ for composable non-identity f, g .

Biequivalences

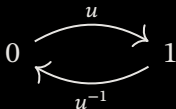
Definition (Campbell 2020, Def. 4.1(1); Johnson--Yau 2021, §6.2) A normal pseudofunctor $F : A \rightarrow B$ is a **biequivalence** if (i) every $b \in B$ is equivalent to some Fa , and (ii) every $F : A(a, a') \rightarrow B(Fa, Fa')$ is an equivalence of categories.

Theorem

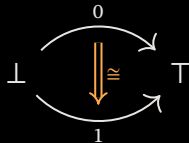
- (1) *Whitehead*: biequivalence $\iff$ pseudo-inverse (Johnson–Yau 2021, Thm. 7.4.1).
- (2) *Coherence*: every bicategory is biequivalent to a 2-category (Johnson–Yau 2021, Thm. 8.4.1; Mac Lane–Paré 1985).
- (3) *Strictification*: $2\text{-Cat} \hookrightarrow \text{Bicat}$ has a left adjoint st ; unit and counit are biequivalences (Campbell 2020, §4.8).

Strict pseudo-inverses need not exist (Lack 2002, Ex. 3.1) A : one object, underlying category $\mathbb{Z}$, one invertible 2-cell $m \Rightarrow n$ iff $m - n$ is even. $A \rightarrow \mathbb{Z}/2$ is a biequivalence, but every 2-functor $\mathbb{Z}/2 \rightarrow A$ is trivial: **strict morphisms are too rigid**.

Free-living cells



$\mathbb{I}$ the free-living isomorphism



$\mathbb{I}_2$ the free-living invertible 2-cell

Observation (Campbell 2020, Obs. 4.2) For every bicategory B :

- ▶ normal pseudofunctors $\mathbb{I} \rightarrow B =$ **adjoint equivalences** in B : the data are $f = F(0 \rightarrow 1)$, $g = F(1 \rightarrow 0)$ and $\varphi_{g,f} : gf \cong \mathbb{I}$, $\varphi_{f,g} : fg \cong \mathbb{I}$; associativity for (f, g, f) and $(g, f, g) =$ the triangle identities;
- ▶ normal pseudofunctors $\mathbb{I}_2 \rightarrow B =$ **invertible 2-cells** of B (no composable pairs in $\mathbb{I}_2$).

These two shapes generate Lack's trivial cofibrations (§5); their nerves $J = N(\mathbb{I})$ and J_2 appear in conditions (i) and (iii) of §6.

§ 3

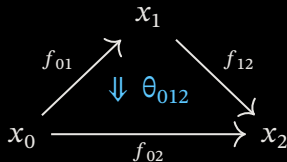
The Duskin nerve

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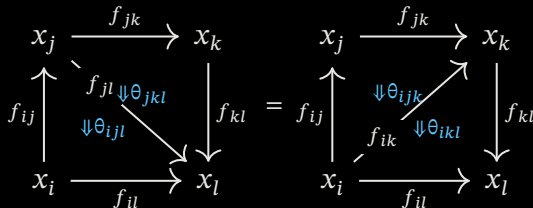
The Duskin nerve

Definition (Street 1996, §10; Duskin 2002, §6; Johnson--Yau 2021, Def. 5.4.7; Kerodon, Tag 009U) With $\mathbf{Bicat}_{\text{nl}}$ = bicategories and normal lax functors, the **Duskin nerve** is $N^{\mathbf{D}} : \mathbf{Bicat}_{\text{nl}} \rightarrow \mathbf{sSet}$, $(N^{\mathbf{D}}\mathbf{B})_n = \{\text{normal lax functors } [n] \rightarrow \mathbf{B}\}$.

a 2-simplex: an **arbitrary** 2-cell



a 3-simplex: an equation between two pastings (Johnson--Yau 2021, Prop. 5.4.12)



Degeneracies: $s_0 f = \mathbf{r}_f$, $s_1 f = \mathbf{l}_f$. $N^{\mathbf{D}}\mathbf{B}$ is 3-coskeletal (3-simplices are *properties* of their boundary).

$N^{\mathbf{D}}\mathbf{C} = \mathbf{NC}$ for a category $\mathbf{C}$.

What works

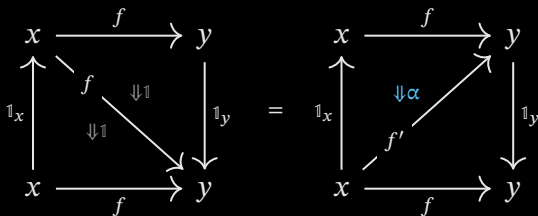
Theorem (Duskin)

- (1) $N^D : \mathbf{Bicat}_{\text{nl}} \rightarrow \mathbf{sSet}$ is **fully faithful** (Kerodon, Thm. 2.3.4.1, Tag 00AU); **normal pseudofunctors** correspond to the maps preserving the **thin** 2-simplices (those with invertible θ) (Kerodon, Cor. 2.3.4.5, Tag 00AX).
- (2) Its image is characterised: $\text{Cosk}_3 X = X \subseteq \text{Cosk}_2 X$, degenerate 2-simplices “invertible”, every $\Lambda_1^2 \rightarrow X$ has an “invertible” filler (Duskin 2002, Thm. 8.6).
- (3) $N^D B$ is a quasi-category **iff every 2-cell of B is invertible** (Duskin 2002, Thm. 8.6; Kerodon, Thm. 2.3.2.1, Tag 00AC).

What fails: an unfillable inner horn

$B = [1; 1]$, the free-living 2-cell $\alpha : f \Rightarrow f' : x \rightarrow y$. A horn $\Lambda_1^3 \rightarrow N^D B$ on the vertices x, x, y, y :

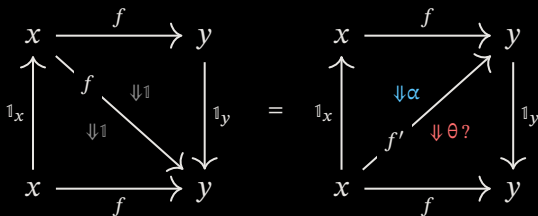
- ▶ $d_3 = (\mathbb{1}_x, f; f')$ filled by α , $d_0 = (f, \mathbb{1}_y; f)$ and $d_2 = (\mathbb{1}_x, f; f)$ filled by identities;
- ▶ the missing face d_1 is a 2-cell $\theta : f' \Rightarrow f$, subject to the cocycle condition:



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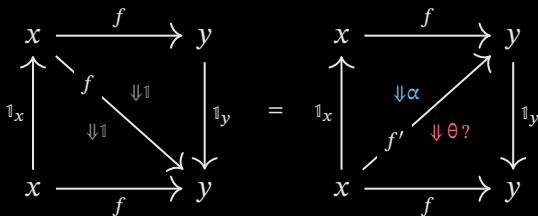
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- ▶ the missing face d_1 is a 2-cell $\theta: f' \Rightarrow f$, subject to the cocycle condition:



Conclusion The equation reads $\theta \cdot \alpha = \mathbb{1}_f$: a filler is a **retraction of α** , and none exists in $[1; 1]$. So $N^D[1; 1]$ is **not** a quasi-category.

Diagnosis, and two cures

- In a quasi-category a 2-simplex is a **witness of composition**, invertible up to homotopy. The Duskin nerve stores *arbitrary* 2-cells in triangles: genuine 2-cells and composition witnesses have the **same shape**.

(a) Keep Δ , add structure **Mark** the thin simplices: (pre)complicial sets (Street 1987, §5; Verity 2008, Thm. 266; Gurski 2009; Ozornova–Rovelli 2021a, Constr. 4.8, Thm. 5.2).

(b) Change the shapes In Θ_2 a 2-cell has its own shape $[1; 1]$; triangles $[2; 0, 0]$ are reserved for **invertible** 2-cells. **This is the coherent nerve**.

The two are compatible (Campbell 2020, Ex. 7.6): with $B^\approx = \text{all objects and 1-cells, invertible 2-cells only}$,

$$\tau^*(NB) = N^D(B^\approx),$$

a quasi-category by Duskin's theorem (3).

$$\begin{array}{ccc} \mathbf{Bicat} & \xrightarrow{N} & [\Theta_2^{\text{op}}, \mathbf{Set}] \\ \downarrow B \mapsto B^\approx & & \downarrow \tau^* \\ \mathbf{Bicat} & \xrightarrow{N^D} & \mathbf{sSet} \end{array}$$

§ 4

The coherent nerve and its full faithfulness

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Θ_2 : the eight cells of degree ≤ 3

$[n; \mathbf{m}] = [n; m_1, \dots, m_n]$ is the free 2-category on n columns, the i -th with $m_i + 1$ parallel 1-cells and m_i stacked 2-cells; $[n; \mathbf{m}](i, j) = [m_{i+1}] \times \dots \times [m_j]$ (posets: **rigid**); degree = $n + \sum m_i$.

$\bullet$
 $[0]$ deg. 0
 an object

$\bullet \xrightarrow{f} \bullet$
 $[1; 0]$ deg. 1
 a 1-cell

$\bullet \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} \bullet$
 $[1; 1]$ deg. 2
 a 2-cell

$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$
 $[2; 0, 0]$ deg. 2
 in NB: $\sigma : gf \cong h$

$\bullet \begin{array}{c} \xrightarrow{\Downarrow \alpha} \\ \Downarrow \beta \\ \xrightarrow{\Downarrow \gamma} \end{array} \bullet$
 $[1; 2]$ deg. 3
 $\gamma = \beta \cdot \alpha$

$\bullet \begin{array}{c} \xrightarrow{\Downarrow \alpha} \\ \Downarrow \beta \\ \xrightarrow{\Downarrow \gamma} \end{array} \bullet \xrightarrow{h} \bullet$
 $[2; 1, 0]$ deg. 3
 whiskering $h\alpha$

$\bullet \xrightarrow{f} \bullet \begin{array}{c} \xrightarrow{\Downarrow \beta} \\ \Downarrow \gamma \\ \xrightarrow{\Downarrow \delta} \end{array} \bullet$
 $[2; 0, 1]$ deg. 3
 whiskering βf

$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$
 $[3; 0, 0, 0]$ deg. 3
 in NB: the pentagon

These eight objects form Θ_b (with fewer morphisms), the subcategory that will turn out to be dense.

The coherent nerve

Lemma (Campbell 2020, §3.4) A normal pseudofunctor between 2-categories with **rigid** codomain is a 2-functor. Hence $\Theta_2 \hookrightarrow 2\text{-Cat} \hookrightarrow \mathbf{Bicat}$ is **fully faithful**.

Proof: the constraints $\varphi_{g,f}$ are invertible 2-cells of a rigid 2-category, hence identities.

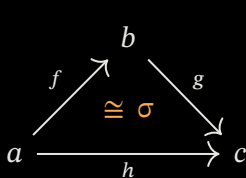
Definition (Leinster 2002, Def. J; Campbell 2020, Def. 3.5) The **coherent nerve** $N : \mathbf{Bicat} \rightarrow [\Theta_2^{\text{op}}, \mathbf{Set}]$ is the singular functor of $\Theta_2 \hookrightarrow \mathbf{Bicat}$:

$$(NB)_{n(m)} = \mathbf{Bicat}([n; m], B), \quad x \cdot (\phi; f) = x \circ (\phi; f), \quad (NF)(x) = F \circ x.$$

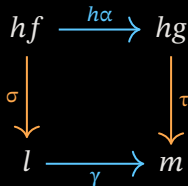
- ▶ $N[n; m] = \Theta_2[n; m]$: representables are nerves. If A is rigid, $NA = N_s A$.
- ▶ $N_s A \subseteq NA$ for every 2-category A ; a weak equivalence (Theorem C, Talk 5).
- ▶ $J_2 = N_s(\mathbb{I}_2) \neq N(\mathbb{I}_2)$: the latter has triangles filled by the invertible 2-cell (Campbell 2020, Not. 4.11).

The coherent nerve in low dimensions

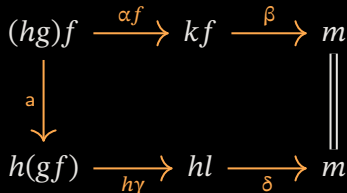
Elements of NB (Campbell 2020, §3.7 and Fig. B): $[0]$ objects, $[1; 0]$ 1-cells, $[1; 1]$ 2-cells $\alpha: f \Rightarrow g$, $[2; 0, 0]$ invertible 2-cells $\sigma: gf \cong h$; in degree 3, equations:



triangle $[2; 0, 0]$



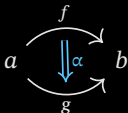
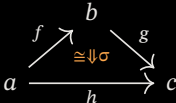
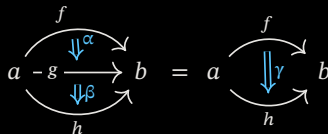
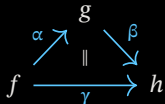
square $[2; 1, 0]: \tau \cdot (h\alpha) = \gamma \cdot \sigma$



pentagon $[3; 0, 0, 0]: \beta \cdot (\alpha f) = \delta \cdot (h\gamma) \cdot a$

- ▶ $[1; 2]: \gamma = \beta \cdot \alpha$; $[2; 0, 1]: \tau \cdot (\beta f) = \gamma \cdot \sigma$ (the mirror square).
- ▶ The $[0], [1; 0], [1; 1]$ part is the underlying reflexive 2-graph; the degeneracies $[2; 0, 0] \rightarrow [1; 0]$ give l_f, r_f .
- ▶ Compared with N^D : non-invertible 2-cells only in the shapes $[1; m]$; triangles carry only invertible ones.

The coherent nerve in low dimensions, after Campbell's Fig. B (I)

$[n; \mathbf{m}]$	$[n; \mathbf{m}] \rightarrow \mathbf{B}$ as a pasting diagram (Campbell 2020, §3.7, Fig. B)	commutative diagram
$[0], [1; 0]$	$a, \quad a \xrightarrow{f} b$	$a, \quad f$
$[1; 1]$		$f \xrightarrow{\alpha} g$
$[2; 0, 0]$		$gf \xrightarrow[\cong]{\sigma} h$
$[1; 2]$		

The coherent nerve in low dimensions, after Campbell's Fig. B (II)

$[2; 1, 0]$

$$\begin{array}{ccc}
 hf & \xrightarrow{h\alpha} & hg \\
 \sigma \downarrow \cong & \parallel & \cong \downarrow \tau \\
 l & \xrightarrow{\gamma} & m
 \end{array}$$

$[2; 0, 1]$

$$\begin{array}{ccc}
 hf & \xrightarrow{\beta f} & kf \\
 \sigma \downarrow \cong & \parallel & \cong \downarrow \tau \\
 l & \xrightarrow{\gamma} & m
 \end{array}$$

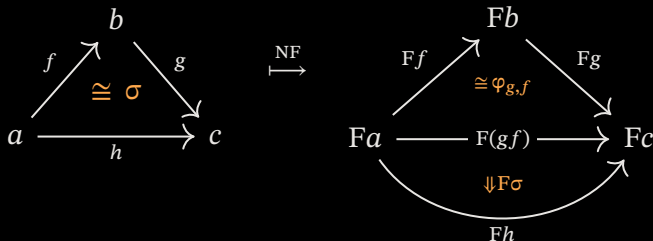
$[3; 0, 0, 0]$

$$\begin{array}{ccc}
 (hg)f & \xrightarrow{\alpha f} & kf \\
 a \downarrow & \parallel & \searrow \beta \\
 h(gf) & \xrightarrow{h\gamma} & hl \\
 & & \nearrow \delta \\
 & & m
 \end{array}$$

Functoriality, and sanity checks

For a normal pseudofunctor $F : A \rightarrow B$, NF is F on $[0]$, $[1; 0]$, $[1; 1]$, and on triangles (Campbell 2020, (3.9)):

$$(NF)(\sigma : gf \cong h) = F\sigma \cdot \varphi_{g,f} : FgFf \Rightarrow F(gf) \Rightarrow Fh.$$

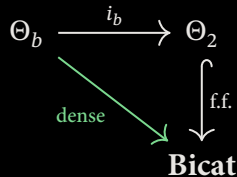


So $NF(\text{commutative triangle } (f, g; gf; \mathbb{1})) = \varphi_{g,f}$: **the constraints of F are visible in NF .**

Full faithfulness: the strategy

Theorem (Campbell 2020, Thm. 3.18) $N : \mathbf{Bicat} \rightarrow [\Theta_2^{\text{op}}, \mathbf{Set}]$ is **fully faithful**, and $N \cong (i_b)_* \circ N_b$.

Lemma (Campbell 2020, Lem. 3.11; Lack--Paoli 2008, Prop. 1.1; Kelly 1982, Thm. 5.13) If G is fully faithful and GF is dense, then G is dense and $\mathcal{C}(G, \mathbb{1}) \cong F_* \circ \mathcal{C}(GF, \mathbb{1})$.



Definition (Campbell 2020, §3.13) $\Theta_b \subset \Theta_2$: **all** morphisms between objects of degree ≤ 2 , and the **monomorphisms** from degree 2 to degree 3. $N_b := i_b^* N : \mathbf{Bicat} \rightarrow [\Theta_b^{\text{op}}, \mathbf{Set}]$.

- ▶ **Prop. 3.14:** N_b is fully faithful, i.e. $\Theta_b \rightarrow \mathbf{Bicat}$ is dense. Then apply the lemma.
- ▶ Why only monos into degree 3: degree-3 elements are **equations**; only their faces carry information.

Proof of Prop. 3.14: faithful, and Steps 1--2

Faithful A normal pseudofunctor F is recovered from $N_b F$: cells from the components at $[0], [1; 0], [1; 1]$, and

$$\varphi_{g,f} = N_b F(\text{commutative triangle } (f, g; gf; \mathbb{1}_{gf})).$$

Full Let $\mathbb{F} : N_b A \rightarrow N_b B$ be natural. We build F with $N_b F = \mathbb{F}$.

- 1 Cells.** The components at $[0], [1; 0], [1; 1]$ give F on objects, 1-cells and 2-cells. Naturality for the faces and degeneracies: sources, targets and identities are preserved, and $F(\beta \cdot \alpha) = F\beta \cdot F\alpha$ **because a $[1; 2]$ -element is a composite**. So each $F : A(a, b) \rightarrow B(Fa, Fb)$ is a functor.
- 2 Constraints.** $\varphi_{g,f} := \mathbb{F}(f, g; gf; \mathbb{1}_{gf}) : FgFf \cong F(gf)$; the boundary is right by naturality for $\delta^0, \delta^1, \delta^2 : [1; 0] \rightarrow [2; 0, 0]$.

Step 3: the key identity

Key identity For every invertible 2-simplex $\sigma: gf \cong h$ of A :

$$\mathbb{F}(\sigma) = F\sigma \cdot \varphi_{g,f}$$

Apply $\mathbb{F}$ to the $[2; 1, 0]$ -element of NA with 2-cell $\mathbb{1}_f$, 1-cell g , triangles $\mathbb{1}_{gf}$, σ , and 2-cell σ (its faces lie in Θ_b):

$$\begin{array}{ccc} gf & \xrightarrow{g\mathbb{1}_f} & gf \\ \mathbb{1}_{gf} \downarrow & & \downarrow \sigma \\ gf & \xrightarrow{\sigma} & h \end{array}$$

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$$\begin{array}{ccc}
 gf & \xrightarrow{g\mathbb{1}_f} & gf \\
 \mathbb{1}_{gf} \downarrow & & \downarrow \sigma \\
 gf & \xrightarrow{\sigma} & h
 \end{array}
 \xrightarrow{\mathbb{F}}
 \begin{array}{ccc}
 Fg Ff & \xrightarrow{Fg \mathbb{1}_{Ff}} & Fg Ff \\
 \varphi_{g,f} \downarrow & & \downarrow \mathbb{F}(\sigma) \\
 F(gf) & \xrightarrow{F\sigma} & Fh
 \end{array}$$

Step 3: the key identity

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Apply $\mathbb{F}$ to the $[2; 1, 0]$ -element of NA with 2-cell $\mathbb{1}_f$, 1-cell g , triangles $\mathbb{1}_{gf}$, σ , and 2-cell σ (its faces lie in Θ_b):

$$\begin{array}{ccc}
 gf & \xrightarrow{g\mathbb{1}_f} & gf \\
 \mathbb{1}_{gf} \downarrow & & \downarrow \sigma \\
 gf & \xrightarrow{\sigma} & h
 \end{array}
 \xrightarrow{\mathbb{F}}
 \begin{array}{ccc}
 Fg Ff & \xrightarrow{Fg \mathbb{1}_{Ff}} & Fg Ff \\
 \varphi_{g,f} \downarrow & & \downarrow \mathbb{F}(\sigma) \\
 F(gf) & \xrightarrow{F\sigma} & Fh
 \end{array}$$

The image is a $[2; 1, 0]$ -element of NB ; its relation is exactly the key identity. **This is the only trick**: everything else is “read an axiom off a shape of degree 3”.

Steps 4--5: the axioms are shapes of degree 3

- *Units* $\leftarrow$ degeneracies $\sigma^0, \sigma^1 : [2; 0, 0] \rightarrow [1; 0]$: $\mathbb{F}(l_f) = l_{Ff}$, i.e. by the key identity $F(l_f) \cdot \varphi_{1,f} = l_{Ff}$ (same for r).
- *Naturality of φ* $\leftarrow [2; 1, 0]$ and $[2; 0, 1]$: $\varphi_{g,f'} \cdot (Fg F\alpha) = F(g\alpha) \cdot \varphi_{g,f}$, and in g .
- *Associativity* $\leftarrow [3; 0, 0, 0]$ with triangles $\mathbb{1}_{hg}, \mathbb{1}_{gf}, \mathbb{1}_{h(gf)}$ and $\mathbf{a}_{h,g,f}$; its image is the associativity axiom of F :

$$\begin{array}{ccccc}
 (Fh Fg) Ff & \xrightarrow{\varphi_{h,g} Ff} & F(hg) Ff & \xrightarrow{F(a) \cdot \varphi_{hg,f}} & F(h(gf)) \\
 \downarrow \mathbf{a} & & & & \parallel \\
 Fh (Fg Ff) & \xrightarrow{Fh \varphi_{g,f}} & Fh F(gf) & \xrightarrow{\varphi_{h,gf}} & F(h(gf))
 \end{array}$$

- *Step 5*: $N_b F = \mathbb{F}$: they agree in degree ≤ 2 (construction and key identity), and degree-3 elements are determined by their faces.

Pattern bicategory axioms = naturality of $\mathbb{F}$ at the shapes of degree 3 (as in Lack–Paoli 2008, §3).

§ 5

Lack's model structure for bicategories

1. Motivation, state of the art, and statements
2. Recollections on bicategories
3. The Duskin nerve
4. The coherent nerve and its full faithfulness
- 5. Lack's model structure for bicategories**
6. The Quillen adjunction (Theorem 5.10)
7. Summary and outlook

Why strict morphisms

- ▶ **Bicat** is **neither complete nor cocomplete**: $\{1\} \rightarrow \{0 < 1\}$ and $\{1\} \rightarrow \{1 < 2\}$ have no pushout among 2-categories and pseudofunctors (Lack 2002, Ex. 4.5). So **N** has **no left adjoint** on **Bicat**: its value at $I[2; 0, 0]$ would represent “composable pairs of 1-cells” (Campbell 2020, Rem. 5.3).
- ▶ **Bicat_s** is **locally finitely presentable**, and **Bicat_s** $\hookrightarrow$ **Bicat** has a left adjoint **Q**, the **normal pseudofunctor classifier** (Lack–Paoli 2008, §5; Blackwell–Kelly–Power 1989, Thm. 3.13):

$$\mathbf{Bicat}_s(\mathbf{Q}A, B) \cong \mathbf{Bicat}(A, B).$$

Unit $A \rightarrow \mathbf{Q}A$ and counit $\mathbf{Q}A \rightarrow A$ are bijective-on-objects biequivalences (Campbell 2020, §4.3).

Q[1; 0] The free bicategory on an arrow: 1-cells $f, f\mathbb{1}, \mathbb{1}f, (\mathbb{1}f)\mathbb{1}, \dots$, all uniquely isomorphic. It is *not* **[1; 0]**: a strict morphism $[1; 0] \rightarrow B$ picks a 1-cell u with $u\mathbb{1} = u = \mathbb{1}u$ and trivial unitors.

Lack's model structure

Warm-up: on $\mathbf{Cat}$, equivalences / injective on objects / isofibrations; generating trivial cofibration $\mathbf{1} \rightarrow \mathbb{I}$ (Lack 2002, Ex. 1.1).

Definition (Campbell 2020, Def. 4.1; Lack 2004, §2) A normal pseudofunctor $F : A \rightarrow B$ is

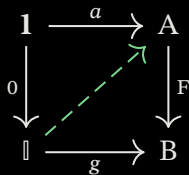
- (1) a **biequivalence** (as in §2);
- (2) an **equifibration** if (i) every equivalence $Fa \rightarrow b$ lifts to an equivalence $a \rightarrow a'$, and (ii) every $F : A(a, a') \rightarrow B(Fa, Fa')$ is an isofibration;
- (3) a **trivial fibration** if it is surjective on objects and every $F : A(a, a') \rightarrow B(Fa, Fa')$ is surjective on objects, full and faithful.

Trivial fibration $\iff$ biequivalence + equifibration.

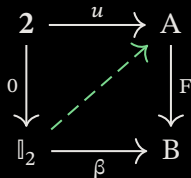
Theorem (Lack 2004, Thm. 3; Campbell 2020, §4.5) There is a cofibrantly generated model structure on $\mathbf{Bicat}_s$ whose weak equivalences, fibrations and trivial fibrations are the strict morphisms that are **biequivalences**, **equifibrations** and **trivial fibrations**.

Anatomy of Lack's model structure

- ▶ Every bicategory is **fibrant**; cofibrant = retract of free (Lack 2004, Lem. 8).
- ▶ $QA \rightarrow A$ is a **cofibrant replacement** (Campbell 2020, §4.5): *strict morphisms out of* QA = *normal pseudofunctors out of* A .
- ▶ Equifibrations = right lifting property in **Bicat** against $\mathbf{1} \rightarrow \mathbb{I}$ and $\mathbf{2} \rightarrow \mathbb{I}_2$ (Campbell 2020, Obs. 4.2):



an adjoint equivalence $Fa \simeq b$ lifts



an invertible $\beta : Fu \cong v$ lifts

- ▶ Homotopy classes of maps = pseudofunctors up to pseudonatural equivalence (Lack 2002, Thm. 5.1, Prop. 4.2, Thm. 4.8): the homotopy theory **sees weak functors** although the category has only strict ones.

2-categories and normal strictification

$$\begin{array}{ccccc} & & \text{st} = LQ & & \\ & \swarrow & & \searrow & \\ \mathbf{2}\text{-Cat} & \xleftarrow{L} & \mathbf{Bicat}_s & \xleftarrow{Q} & \mathbf{Bicat} \\ & \searrow \perp & & \swarrow \perp & \\ & \mathbf{Bicat}_s & & \mathbf{Bicat} & \end{array}$$

Theorem (Lack 2004, Thms. 4 and 11, Lem. 10; Campbell 2020, §§4.6--4.8)

- (1) $\mathbf{2}\text{-Cat}$ has the model structure right-induced from $\mathbf{Bicat}_s$, and $L \dashv \text{incl}$ is a **Quillen equivalence**.
- (2) For a **cofibrant** bicategory A , the unit $A \rightarrow LA$ is a biequivalence. Hence a strict morphism between cofibrant bicategories is a biequivalence iff L sends it to one.
- (3) $\text{st} = L \circ Q$ is the normal strictification.

Statement (2) is used in Lemma 5.8 (§6).

§ 6

The Quillen adjunction (Theorem 5.10)

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A recognition principle for left Quillen functors

Proposition (Campbell 2020, Prop. 4.13) Let $F : [\Theta_2^{\text{op}}, \mathbf{Set}] \rightarrow \mathcal{M}$ be cocontinuous, sending monomorphisms to cofibrations. Then F sends Ara's weak equivalences to weak equivalences iff it sends to weak equivalences

- (i) the projections $J \times \Theta_2[n; \mathbf{m}] \rightarrow \Theta_2[n; \mathbf{m}]$;
- (ii) the spine inclusions $I[n; \mathbf{m}] \rightarrow \Theta_2[n; \mathbf{m}]$;
- (iii) $j_2 : J_2 \rightarrow \Theta_2[1; 0]$.

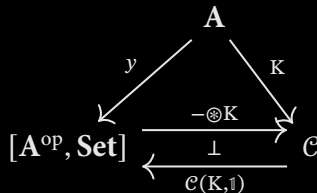
Then F is **left Quillen**.

- ▶ Ara's model structure: the Cisinski model structure generated by (ii) and (iii) (Ara 2014, §5.17).
- ▶ Proof: $W_F = \{f : F(f) \text{ weak equivalence}\}$ is a Θ_2 -localisateur (Cisinski 2006, Déf. 8.2.10); it contains the trivial fibrations by (i) (Cisinski 2006, Prop. 8.2.15), so it contains Ara's localiser.
- ▶ (i) can be reduced to the single map $J \rightarrow \Theta_2[0]$ (Maehara 2020, Lem. 5.1; Campbell 2020, Rem. 4.14).

Kan's construction

Recall (Kan 1958, §14; Campbell 2020, Rec. 3.1) For $K: \mathbf{A} \rightarrow \mathcal{C}$ ($\mathbf{A}$ small), the **singular functor** $\mathcal{C}(K, 1): \mathcal{C} \rightarrow [\mathbf{A}^{\text{op}}, \mathbf{Set}]$ sends C to $\mathcal{C}(K-, C)$; K is **dense** if $\mathcal{C}(K, 1)$ is fully faithful. If $\mathcal{C}$ is **co-complete**, $\mathcal{C}(K, 1)$ has a left adjoint, the left Kan extension of K along the Yoneda embedding y :

$$F \otimes K \cong \int^{\mathbf{A}} F\mathbf{A} \times K\mathbf{A} \cong \text{colim}_{y\mathbf{A} \rightarrow F} K\mathbf{A}.$$



$y\mathbf{A} \otimes K \cong K\mathbf{A}$: a map $F \otimes K \rightarrow C$ is a compatible family of maps $K\mathbf{A} \rightarrow C$, one for each element of F .

K	singular functor	left adjoint
$\Delta \rightarrow \mathbf{Top}$	Sing	geometric realisation (Kan 1958, §1)
$\Delta \hookrightarrow \mathbf{Cat}$	nerve N	τ_1 (Lurie, HTT, Prop. 1.2.3.1)
$\mathcal{O}_2 \hookrightarrow \mathbf{2-Cat}$	strict N_S (Talk 3)	exists, as $\mathbf{2-Cat}$ is cocomplete (Ara 2014, §5.8)
$\mathcal{O}_2 \hookrightarrow \mathbf{Bicat}$	coherent N , dense (Thm. 3.18)	none ; $\mathbf{Bicat}$ is not cocomplete (Campbell 2020, Rem. 5.3)
$[n; m] \mapsto Q[n; m] \in \mathbf{Bicat}_s$	$N _{\mathbf{Bicat}_s}$	τ_b , next slide (Campbell 2020, (5.2))

The adjunction

For $B \in \mathbf{Bicat}_s$: $(NB)_{n(m)} = \mathbf{Bicat}([n; \mathbf{m}], B) \cong \mathbf{Bicat}_s(Q[n; \mathbf{m}], B)$, so by Kan's construction (Campbell 2020, (5.2))

$$[\Theta_2^{\text{op}}, \mathbf{Set}] \begin{array}{c} \xrightarrow{\tau_b} \\ \perp \\ \xleftarrow{N} \end{array} \mathbf{Bicat}_s \quad \tau_b X = \operatorname{colim}_{\Theta_2[n; \mathbf{m}] \rightarrow X} Q[n; \mathbf{m}]$$

and, composing with $L \dashv \text{incl}$, $\tau_2 := L\tau_b \dashv N : \mathbf{2-Cat} \rightleftarrows [\Theta_2^{\text{op}}, \mathbf{Set}]$ (Campbell 2020, (5.5)).

Observation (Campbell 2020, Obs. 5.4) $\tau_b \circ N \cong Q$ and $\tau_2 \circ N \cong \text{st}$. Hence the counits are biequivalences.

$$\begin{array}{ccc} \mathbf{Bicat} & \xrightarrow{N} & [\Theta_2^{\text{op}}, \mathbf{Set}] \\ & \searrow Q \cong & \downarrow \tau_b \\ & & \mathbf{Bicat}_s \end{array}$$

$$\begin{array}{ccc} \mathbf{Bicat} & \xrightarrow{N} & [\Theta_2^{\text{op}}, \mathbf{Set}] \\ & \searrow \text{st} \cong & \downarrow \tau_2 \\ & & \mathbf{2-Cat} \end{array}$$

Proof: $\mathbf{Bicat}_s(\tau_b NA, B) \cong [\Theta_2^{\text{op}}, \mathbf{Set}](NA, NB) \cong \mathbf{Bicat}(A, B) \cong \mathbf{Bicat}_s(QA, B)$, the middle step being **Thm. 3.18**.

τ_b preserves cofibrations

Proposition (Campbell 2020, Prop. 5.6) N preserves and reflects trivial fibrations. Hence τ_b sends monomorphisms to **cofibrations**.

$$\begin{array}{ccc}
 \partial\Theta_2[n; \mathbf{m}] & \longrightarrow & \mathbf{NA} \\
 \downarrow & \nearrow \text{dashed green arrow} & \downarrow \text{NF} \\
 \Theta_2[n; \mathbf{m}] & \longrightarrow & \mathbf{NB}
 \end{array}$$

by $N \cong (i_b)_* N_b$: only $\deg \leq 3$
matter

shape

$\deg \geq 4$

$[0], [1; 0], [1; 1],$
 $[1; 2]$

$[2; 0, 0]$

$[2; 1, 0], [2; 0, 1],$
 $[3; 0, 0, 0]$

the lifting property says

nothing: $i_b^* \delta_{n(m)}$ is invertible

surjective on objects, locally surjective on objects,
locally full, locally faithful: **a trivial fibration**

take σ with $F\sigma = \sigma' \cdot \varphi_{g,f}^{-1}$; invertible since locally
f.f. functors reflect isos

equations: they lift by local faithfulness

By adjunction, $\tau_b(\delta_{n(m)})$ has the LLP against trivial fibrations; monos are generated by the $\delta_{n(m)}$.

τ_b preserves weak equivalences

Observation (Campbell 2020, Obs. 5.7) $F \dashv U : \mathbf{2-Cat} \rightleftarrows [\mathbb{G}_2^{\text{op}}, \mathbf{Set}]$ is the composite of $\tau_2 \dashv N$ and $g_! \dashv g^*$. Hence the spine inclusion $I[n; \mathbf{m}] \rightarrow \Theta_2[n; \mathbf{m}] = N[n; \mathbf{m}]$ is the **unit** of $\tau_2 \dashv N$ at $I[n; \mathbf{m}]$.

Lemma (Campbell 2020, Lem. 5.8) τ_b sends every unit component $\eta_X : X \rightarrow N\tau_2 X$ to a biequivalence.

$\tau_b X$ is cofibrant (Prop. 5.6), so by Lack 2004 it suffices that $L\tau_b(\eta_X) = \tau_2(\eta_X)$ is one: triangle identity $\varepsilon_{\tau_2 X} \circ \tau_2(\eta_X) = 1$, with ε a biequivalence (Obs. 5.4); 2-out-of-3.

Proposition (Campbell 2020, Prop. 5.9) τ_b sends Ara's weak equivalences to **biequivalences**. Check (i)–(iii):

- (i) $J \times \Theta_2[n; \mathbf{m}] \rightarrow \Theta_2[n; \mathbf{m}]$ is N of $\mathbb{I} \times [n; \mathbf{m}] \rightarrow [n; \mathbf{m}]$; by Obs. 5.4, τ_b sends it to Q of a biequivalence;
- (ii) spines are unit components: Lemma 5.8;
- (iii) $\tau_b(j_2) \cong Q(\mathbb{I}_2 \rightarrow 2)$, a biequivalence (maps $J_2 \rightarrow \mathbf{NB} = \text{invertible 2-cells}$).

The theorem

Theorem (Campbell 2020, Thm. 5.10) $\tau_b \dashv N : \mathbf{Bicat}_s \rightleftarrows [\Theta_2^{\text{op}}, \mathbf{Set}]$ is a **Quillen adjunction** between Lack's model structure and Ara's, and $RN : \text{Ho}(\mathbf{Bicat}_s) \rightarrow \text{Ho}([\Theta_2^{\text{op}}, \mathbf{Set}])$ is **fully faithful**.

Proof τ_b preserves cofibrations (Prop. 5.6) and weak equivalences (Prop. 5.9). Every bicategory is fibrant and every Θ_2 -set is cofibrant, so the derived counit at B is $\tau_b NB \cong QB \rightarrow B$, a biequivalence (Obs. 5.4). A right adjoint whose counit is invertible is fully faithful.

Theorem A (Campbell 2020, Cors. 5.11 and 5.13) Every NB is a **2-quasi-category** (right Quillen functors preserve fibrant objects), and $N : \mathbf{Bicat} \rightarrow \mathbf{2-qCat}$ is fully faithful.

- ▶ N **creates** Lack's weak equivalences and fibrations: right-induced (Campbell 2020, Obs. 5.15).
- ▶ **Not** a Quillen equivalence: the homs of nerves are 1-truncated. Talk 5: localise at $\partial\Theta_2[1; 3] \rightarrow \Theta_2[1; 3]$ (Campbell 2020, Thm. 8.7).

§ 7

Summary and outlook

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Summary

Today

- ▶ $N : \mathbf{Bicat} \rightarrow [\Theta_2^{\text{op}}, \mathbf{Set}]$ is fully faithful and determined in degree ≤ 3 (Campbell 2020, Thm. 3.18).
- ▶ $\tau_b \dashv N$ is Quillen with $\mathbf{RN}$ fully faithful (Campbell 2020, Thm. 5.10).
- ▶ Nerves of bicategories are 2-quasi-categories (Theorem A); Lack's model structure is right-induced from Ara's (Campbell 2020, Obs. 5.15).

The conceptual gain Bicatogories, with normal pseudofunctors, sit inside Ara's 2-quasi-categories **exactly as categories sit inside quasi-categories**: fully faithfully and homotopically fully faithfully. The price was to change the shapes (simplices $\rightarrow \Theta_2$ -cells) and the functors (strict $\rightarrow$ normal pseudo); the only real input is the **density of the eight cells of degree ≤ 3** .

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All other references (with theorem numbers) are in the notes, `talk4_campbell_notes.pdf`.